Completing The Squares

Developed as a series of exercises

Section 1.5 of the textbook (pg.48) covers an equation-solving method called *completing the square*. Not only is it used in solving quadratic equations, it is also immensely useful in graphing quadratic functions (Chapter 2) and computing certain integrals (Calculus). It is, therefore, imperative that we master it thoroughly.

Exercise 1 Why do we add "the square of the half of the coefficent of the linear term"? Where did we get this absurd number? Expand $(x+p)^2$, and compare the coefficient of x with the constant term.
Solution. $(x+p)^2 = x^2 + 2px + p^2$. Note that $p^2 = ((2p) \div 2)^2$. Also note that p in $(x+p)^2$ is one-half of the coefficient of x.

Exercise 2 Solve $x^2 + 4x = -4$ by completing the square. Check your answer using the perfect square formula.
Solution. $x^2 + 4x = -4$
$\iff x^2 + 4x + (4 \div 2)^2 = -4 + (4 \div 2)^2$
$\iff x^2 + 4x + 4 = 0$
$\iff (x+2)^2 = 0$
$\iff x = -2$

Exercise 3 Solve $t^2 - 6t = -8$ by completing the square. Check your answer by factoring.
Solution. $t^2 - 6t = -8$
$\iff t^2 - 6t + (-6 \div 2)^2 = -8 + (-6 \div 2)^2$
$\iff t^2 - 6t + 9 = -8 + 9$
$\iff (t-3)^2 = 1$
$\iff t - 3 = \pm 1$
$\iff t = 3 \pm 1$
$\iff t = 4$ or $t = 2$

Exercise 4 Solve $w^2 + 2w - 4 = 1$ by completing the square.
Solution. $w^2 + 2w - 4 = 1$
$\iff w^2 + 2w = 1 + 4$
$\iff w^2 + 2w + (2 \div 1)^2 = 1 + 4 + (2 \div 1)^2$
$\iff w^2 + 2w + 1 = 1 + 4 + 1$
$\iff (w+1)^2 = 6$
$\iff w + 1 = \pm\sqrt{6}$
$\iff w = -1 \pm \sqrt{6}$

Exercise 5 Solve $y^2 + y - 5 = 0$ by completing the square.
Solution. $y^2 + y - 5 = 0$
$\iff y^2 + y = 5$
$\iff y^2 + y + (1 \div 2)^2 = 5 + (1 \div 2)^2$
$\iff y^2 + y + \frac{1}{4} = 5 + \frac{1}{4}$

$\iff \left(y+\frac{1}{2}\right)^2 = \frac{21}{4}$
$\iff y+\frac{1}{2} = \sqrt{\frac{21}{4}}$
$\iff y = -\frac{1}{2} + \sqrt{\frac{21}{4}}$
$\iff y = -\frac{1}{2} + \frac{\sqrt{21}}{2}$
$\iff y = \frac{-1+\sqrt{21}}{2}$

Exercise 6 Can we develope a formula for completing the square in the monic case? Given $x^2+bx+c = (x-h)^2+k$, expand the right hand side and express h in terms of b. Also, express k in terms of b and c.
Solution. $x^2+bx+c = (x-h)^2+k = x^2-2hx+h^2+k$, therefore $b=-2h$ and $c=h^2+k$. Solving for the appropriate variable gives $h = -\frac{b}{2}$ and $k = c-h^2 = c-\frac{b^2}{4}$.

Exercise 7 We move on to the non-monic case; that is, when the coefficient of the quadratic term is not 1. To start off, solve $3q^2+18q+20=-7$ by completing the square. Check your answer using the perfect square formula.
Solution. $3q^2+18q+20=-7$
$\iff 3q^2+18q = -20-7$
$\iff q^2+6q = (-20-7)\div 3$
$\iff q^2+6q=-9$
$\iff q^2+6q+(6\div 2)^2 = -9+(6\div 2)^2$
$\iff q^2+6q+9 = -9+9$
$\iff (q+3)^2 = 0$
$\iff q=-3$

Exercise 8 Solve $7z^2-28z+21=0$ by completing the square. Check your answer by factoring.
Solution. $7z^2-28z+21=0$
$\iff 7z^2-28z=-21$
$\iff z^2-4z=-21\div 7$
$\iff z^2-4z=-3$
$\iff z^2-4z+(-4\div 2)^2 = -3+(4\div 2)^2$
$\iff z^2-4z+4=-3+4$
$\iff (z-2)^2=1$
$\iff z-2=\pm 1$
$\iff z=2\pm 1$
$\iff z=3$ or $z=1$

Exercise 9 Solve $3d^2-9d+1=0$ by completing the square.
Solution. $3d^2-9d+1=0$
$\iff 3d^2-9d=-1$

$\iff d^2 - 3d = -\dfrac{1}{3}$
$\iff d^2 - 3d + (-3 \div 2)^2 = -\dfrac{1}{3} + (3 \div 2)^2$
$\iff d^2 - 3d + \dfrac{9}{4} = -\dfrac{1}{3} + \dfrac{9}{4}$
$\iff \left(d - \dfrac{3}{2}\right)^2 = \dfrac{23}{12}$
$\iff d - \dfrac{3}{2} = \pm\sqrt{\dfrac{23}{12}}$
$\iff d = \dfrac{3}{2} \pm \sqrt{\dfrac{23}{12}}$
$\iff d = \dfrac{3}{2} \pm \dfrac{\sqrt{23}\sqrt{3}}{6}$
$\iff d = \dfrac{3}{2} \pm \dfrac{\sqrt{69}}{6}$
$\iff d = \dfrac{9 + \sqrt{69}}{6}$

Exercise 10 Solve $5f^2 + 7f + 2 = 3$ by completing the square.
Solution. $5f^2 + 7f + 2 = 3$
$\iff 5f^2 + 7f = 3 - 2$
$\iff f^2 + \dfrac{7}{5}f = (3 - 2) \div 5$
$\iff f^2 + \dfrac{7}{5}f = \dfrac{1}{5}$
$\iff f^2 + \dfrac{7}{5}f + \left(\dfrac{7}{5} \div 2\right)^2 = \dfrac{1}{5} + \left(\dfrac{7}{5} \div 2\right)^2$
$\iff f^2 + \dfrac{7}{5}f + \dfrac{49}{100} = \dfrac{1}{5} + \dfrac{49}{100}$
$\iff \left(f + \dfrac{7}{10}\right)^2 = \dfrac{69}{100}$
$\iff f + \dfrac{7}{10} = \pm\sqrt{\dfrac{69}{100}}$
$\iff f = -\dfrac{7}{10} \pm \sqrt{\dfrac{69}{100}}$
$\iff f = -\dfrac{7}{10} \pm \dfrac{\sqrt{69}}{10}$
$\iff f = -\dfrac{7 + \sqrt{69}}{10}$

Exercise 11 A general formula for completing the square can also be developed. Given $ax^2 + bx + c = a(x - h)^2 + k$, express h in terms of a and b. Also, express k in terms of a, b, and c.
Solution. $ax^2 + bx + c = a(x - h)^2 + k = a(x^2 - 2hx + h^2) + k = ax^2 - 2ahx + ah^2 + k$, therefore $b = -2ah$ and $c = ah^2 + k$. Solving for the appropriate variable gives $h = -\dfrac{b}{2a}$ and

$k = c - ah^2 = c - a\frac{(-b)^2}{(2a)^2} = c - \frac{b^2}{4a}$.

Remark: What does this tell us? If we have any quadratic expression $ax^2 + bx + c$, we can convert it to the "completed-square" format by the formula

$$ax^2 + bx + c = a(x-h)^2 + k, \text{ where } h = -\frac{b}{2a} \text{ and } k = c - \frac{b^2}{4a}.$$

This is very useful for certain purposes. When solving a general quadratic equation, however, using the quadratic formula is more efficient.

Exercise 12 We are now ready to prove the quadratic formula. Verify that the roots of the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

via solving the equation by completing the square.

Solution. $ax^2 + bx + c = 0$

$\iff ax^2 + bx = -c$

$\iff x^2 + \frac{b}{a} = -c \div a$

$\iff x^2 + \frac{b}{a} = -\frac{c}{a}$

$\iff x^2 + \frac{b}{a} + \left(\frac{b}{a} \div 2\right)^2 = -\frac{c}{a} + \left(\frac{b}{a} \div 2\right)^2$

$\iff x^2 + \frac{b}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$

$\iff \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$

$\iff x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac + b^2}{4a^2}}$

$\iff x = -\frac{b}{2a} \pm \sqrt{\frac{-4ac + b^2}{4a^2}}$

$\iff x = -\frac{b}{2a} \pm \frac{\sqrt{-4ac + b^2}}{2a}$

$\iff x = \frac{-b \pm \sqrt{-4ac + b^2}}{2a}$

$\iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$