Problem 1. Let $f(x) = 3 - 5x + 4x^2$. Find $f(a)$, $f(a+h)$, and the difference quotient

$$\frac{f(a+h) - f(a)}{h},$$

where $h \neq 0$.

Solution. We have $f(a) = 3 - 5a + 4a^2$ and

$$f(a+h) = 3 - 5(a+h) + 4(a+h)^2 = 3 - 5a + 4a^2 - 5h + 8ah + 4h^2,$$

whence

$$\begin{aligned}\frac{f(a+h) - f(a)}{h} &= \frac{(3 - 5a + 4a^2 - 5h + 8ah + 4h^2) - (3 - 5a + 4a^2)}{h} \\ &= \frac{-5h + 8ah + 4h^2}{h} \\ &= -5 + 8a + 4h.\end{aligned}$$

□

Problem 2. Find the domain of

$$g(x) = \frac{\sqrt{2+x}}{3-x}.$$

Solution. There are two restrictions for this function:

(i) The denominator must be nonzero: this yields $3 - x \neq 0$, or $x \neq 3$.

(ii) The number inside the square root must be nonnegative: this yields $2 + x \geq 0$, or $x \geq -2$.

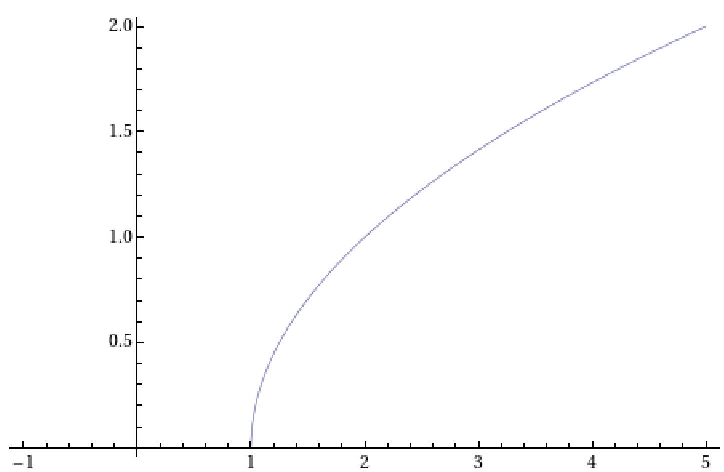
It thus follows that the domain of g is $x \geq -2$ *and* $x \neq 3$, or

$$[-2, 3) \cup (3, \infty).$$

□

Problem 3. Use the graphing calculator to draw the graph of $f(x) = \sqrt{x-1}$, and find the domain and range of f.

Solution. The graph of f is



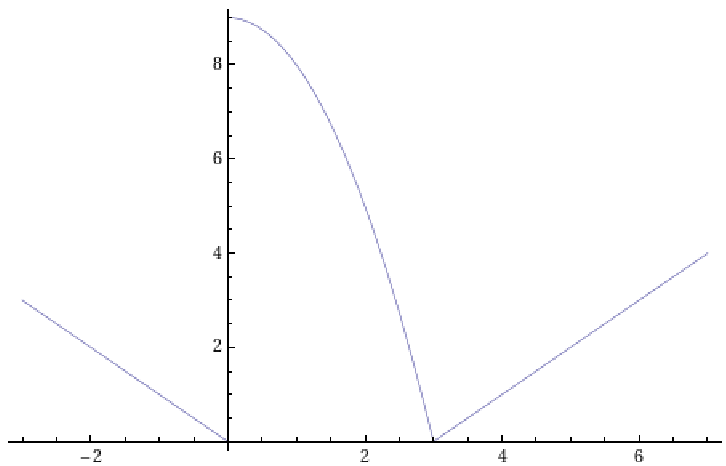
The domain of f is $[1, \infty)$. The range of f is $[0, \infty)$. □

Problem 4. Sketch the graph of

$$f(x) = \begin{cases} -x & \text{if } x \leq 0; \\ 9 - x^2 & \text{if } 0 < x \leq 3; \\ x - 3 & \text{if } x > 3. \end{cases}$$

and compute $f(-1)$, $f(1)$, $f(3)$, $f(5)$.

Solution. The graph of f is



We have

- $f(-1) = -(-1) = 1$.
- $f(1) = 9 - (1)^2 = 8$.
- $f(3) = 9 - (3)^2 = 0$.
- $f(5) = (5) - 3 = 2$.

□

Problem 5. A rancher with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle.

(a) Find a function that models the total area of the four pens.

(b) Find the largest possible total area of the four pens.

Solution. A possible choice of function that models the area of the four pens is

$$A(x) = x \cdot \frac{750 - 5x}{2} = -\frac{5}{2}x^2 + 375x.$$

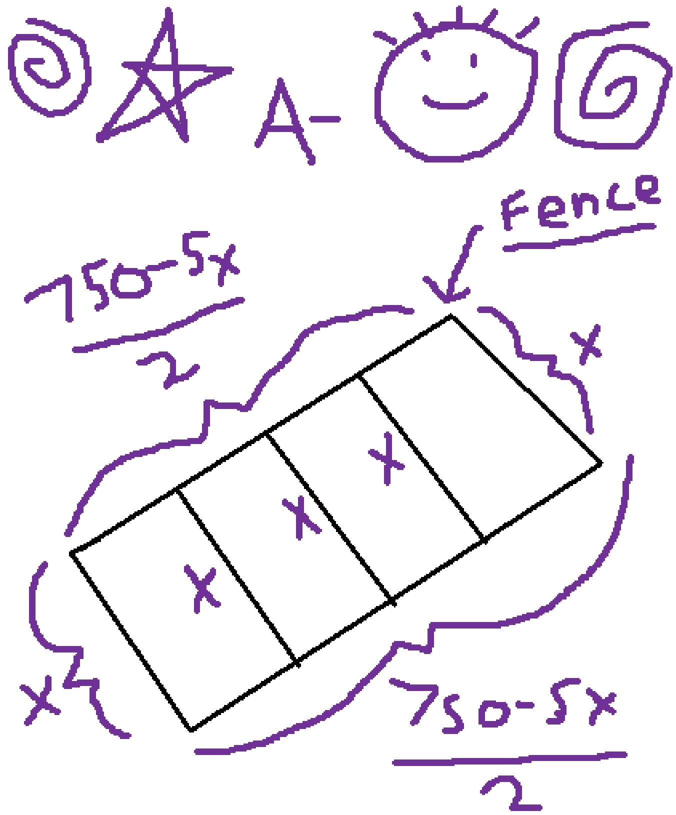


Figure 1: A rancher and his pens. Artwork courtesy of Matthew Inverso.

To simplify the computation, we consider instead the following function

$$B(x) = \frac{2}{5}A(x) = -x^2 + 150x.$$

The function A achieves its maximum at x if and only if B achieves its maximum at x. Why is this? Finding the maximum of B amounts to finding the vertex of B, which requires rewriting B in the form

$$B(x) = a(x-h)^2 + k.$$

In this form, the vertex is at $x = h$, and the maximum is $B(h) = k$. Now, we note that

$$A(x) = \frac{5}{2}B(x) = \frac{5}{2}a(x-h)^2 + \frac{5}{2}k$$

in this form, whence the vertex of A is at $x = h$ as well. It therefore suffices to find the maximum of $B(x)$, and scale it back to find the maximum of $A(x)$.

Completing the squares, we obtain

$$\begin{aligned} B(x) &= -x^2 + 150x \\ &= -x^2 + 150x - \left(\frac{150}{2}\right)^2 + \left(\frac{150}{2}\right)^2 \\ &= -x^2 + 150x - 5625 + 5625 \\ &= -(x^2 - 150x + 5625) + 5625 \\ &= -(x - 75)^2 + 5625. \end{aligned}$$

Therefore, B achieves its maximum at $x = 75$, and its maximum is

$$B(75) = 5625.$$

It thus follows that the maximum value of A is

$$A(75) = \frac{5}{2}B(75) = \frac{28125}{2} = 14062.5 \text{ square feet.}$$

□

Problem 6. An open box with a square base is to have a volume of 12 ft^3.

(a) Find a function that models the surface area of the box.

(b) Find the box dimensions that minimize the amount of material used.

Solution. Consider the following diagram:

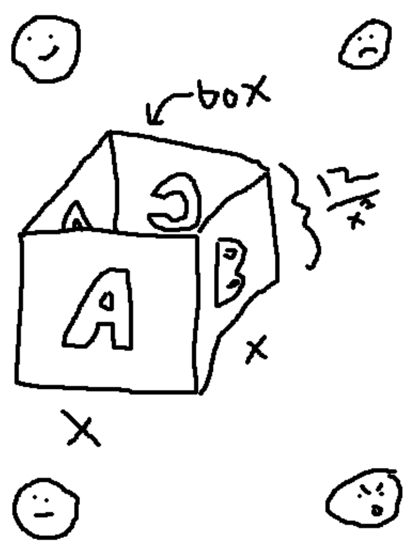
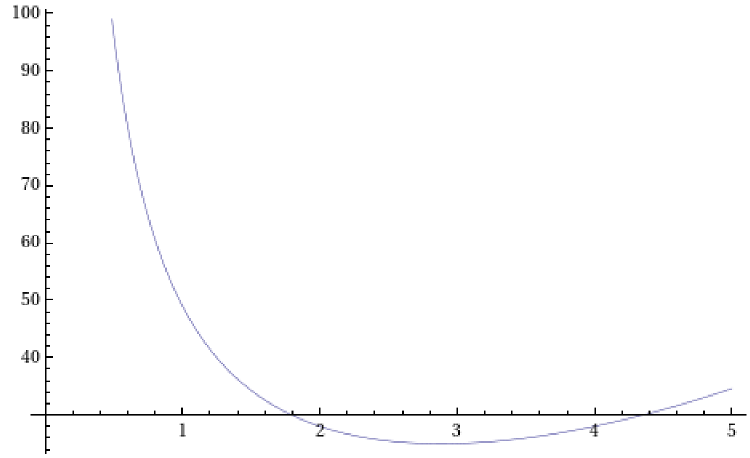


Figure 2: A cube, anxiously waiting to be optimized. Artwork courtesy of Matthew Inverso.

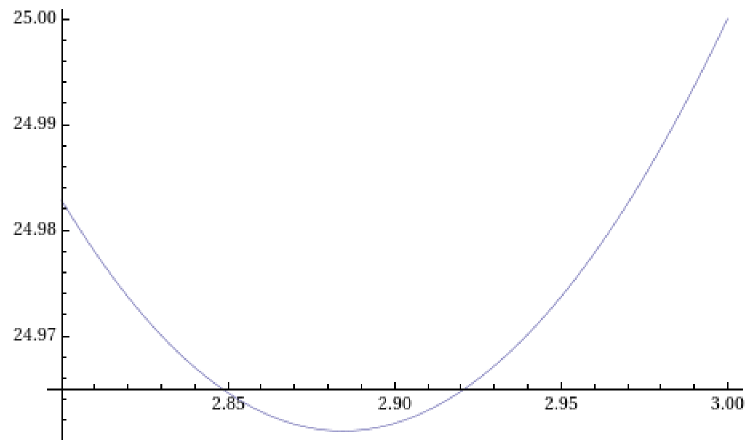
A possible choice of function that models the surface area of the box is

$$\begin{aligned} A(x) &= \text{area of the base} + 4 \times \text{area of the side} \\ &= x^2 + 4 \cdot \frac{12}{x^2} \cdot x \\ &= x^2 + \frac{48}{x}. \end{aligned}$$

Looking at the graph, we see that A achieves the minimum near 3:



A zoom-in shows that A achieves the minimum near 2.88:



(The minimum is achieved at $x = 2\sqrt[3]{3} \approx 2.8845$, but calculating the precise minimum is beyond the scope of this course.)

The approximate dimensions of the box that minimizes the amount of material are

$$2.88 \times 2.88 \times 1.44.$$

$\square$

Problem 7. Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate $(f \circ f)(-1)$ and $(g \circ g)(2)$.

Solution. $f(-1) = 3(-1) - 5 = -8$, and so

$$(f \circ f)(-1) = f(f(-1)) = f(-8) = 3(-8) - 5 = -29.$$

Similarly, $g(2) = 2 - (2)^2 = -2$, and so

$$(g \circ g)(2) = g(g(2)) = g(-2) = 2 - (-2)^2 = -2.$$

□

Problem 8. Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution. $(f \circ g)(x) = 3(2 - x^2) - 5 = 6 - 3x^2 - 5 = -3x^2 + 1$, and $(g \circ f)(x) = 2 - (3x - 5)^2 = 2 - (9x^2 - 30x + 25) = -9x^2 + 30x - 23$. □

Problem 9. Find the inverse function of

$$f(x) = \frac{1 + 3x}{5 - 2x}.$$

Solution. We first write

$$y = \frac{1 + 3x}{5 - 2x}$$

and solve for x:

$$\begin{aligned} y &= \frac{1 + 3x}{5 - 2x} \\ y(5 - 2x) &= 1 + 3x \\ 5y - 2xy &= 1 + 3x \\ 5y - 1 &= 3x + 2xy \\ 5y - 1 &= x(3 + 2y) \\ \frac{5y - 1}{3 + 2y} &= x. \end{aligned}$$

It thus follows that the inverse function of f is

$$f^{-1}(x) = \frac{5x - 1}{2x + 3}.$$

□

Problem 10. Find the inverse function of $f(x) = \sqrt{2 + 5x}$. Find the domain and range of f. Find the domain and range of f^{-1}. Compute $(f^{-1} \circ g)(2)$, where $g(x) = \sqrt{x + 1}$.

Solution. We first write

$$y = \sqrt{2 + 5x}$$

and solve for x:

$$\begin{aligned} y &= \sqrt{2+5x} \\ y^2 &= 2+5x \\ y^2-2 &= 5x \\ \frac{y^2-2}{5} &= x. \end{aligned}$$

It follows that the inverse function of f is

$$f^{-1}(x) = \frac{x^2-2}{5}.$$

We note that the domain of f is $2+5x \geq 0$ or $x \geq -\frac{2}{5}$, whence the range of f^{-1} must be

$$\left[-\frac{2}{5}, \infty\right).$$

The range of f is $[0, \infty)$, and so the domain of f^{-1} is $[0, \infty)$.

Now, $g(2) = \sqrt{2+1} = \sqrt{3}$, and so

$$(f^{-1} \circ g)(2) = f^{-1}(g(2)) = f^{-1}(\sqrt{3}) = \frac{(\sqrt{3})^2-2}{5} = \frac{3-2}{5} = \frac{1}{5}.$$

$\square$