Discrete Analogues in Harmonic Analysis
Course Outline

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Summer Program in Princeton
Analysis and Geometry
2011

This course will survey a relatively new area of harmonic analysis: discrete operators modeled on classical operators, such as singular integral operators, maximal functions, fractional integral operators, and Radon transforms. The classical operators are well understood, but their discrete analogues require new approaches, and in some cases remain rather mysterious. This course will offer a bird's eye view of discrete operators and their classical counterparts and will introduce several techniques (stemming from number theory) for treating discrete operators. The series of six lectures will follow (roughly) this progression of ideas:

Lecture 1: Basic operators in harmonic analysis

Introduce three basic operators in the Euclidean setting of $\mathbb{R}^n$: the Hardy-Littlewood maximal function, the Hilbert transform, and fractional integral operators.

Lecture 2: A first look at discrete convolution operators

Introduce the notion of a discrete operator and prove basic tools for convolution operators on $\mathbb{Z}^n$. Prove results for discrete analogues of the three operators introduced in Lecture 1.

Lecture 3: The Radon paradigm on the discrete side

A brief look at a few examples of discrete fractional integral operators involving Radon-type behavior. Define our main operator of interest: a discrete fractional Radon transform. Derive the main conjecture for this operator. Introduce the circle method.

Lectures 4-5: Application of the circle method

Prove main theorem for key operator, using three tools from number theory: the circle method, exponential sums, and theta functions.

Lecture 6: Further topics

Examine a higher degree version of our key operator, and its relation to problems in number theory. Outline further directions of interest in the study of discrete operators, including a range of open problems.

Preparation for the course

There does not yet exist a textbook on discrete analogues in harmonic analysis, thus the material of this course will be largely self-contained. However, it would be helpful to arrive with a basic familiarity with the relevant operators on $\mathbb{R}^n$, and with an understanding of a few tools from number theory. You do not need to follow any particular texts, but a few options are suggested below. In each case you will be able to get by with the simpler presentations suggested, and you can focus on these if the more advanced texts seem out of reach. More reading material of specific relevance will be made available as we proceed through the course.

Analytic preparation

The basics:

- The Fourier transform: for a simpler introduction, see Chapters 5-6 of [5], or for a more advanced treatment, see Chapter I of [4].
- L^p spaces, ℓ^p spaces, duality, interpolation: the basics are covered in Chapter 6 of [1], and a more advanced treatment (including weak-type L^p spaces and complex interpolation) is given in Chapter V of [4].
- The Hardy-Littlewood maximal function: see Section 3.4 of [1]; or for a more advanced treatment, see Chapter II of [7] or Chapter II §3 of [4].

Advanced topics for ambitious readers:

- The Hilbert transform: this (along with a much more general theory than we need) can be found in Chapter II of [7]. The Hilbert transform is also treated in Chapter VI of [4].
- Fractional integral operators: technically, we will briefly call upon a special case of the Hardy-Littlewood-Sobolev theorem, which is given in Chapter V §1 of [7]. But we will give a simpler treatment in class.

Number theoretic preparation

- Approximation of irrationals by rationals: either the basic Dirichlet's principle in Chapter XI of [3] or (not necessary) the fancier Farey dissection in Chapter III of [3].
- Gauss sums: a fancy treatment is given in §3.4 of Chapter 3 of [2], but the simple classical bound will be covered in many analytic number theory texts.
- The Weyl bound: See §8.2 of Chapter 8 of [2] for a clear proof of the degree 2 case of the Weyl bound, and a discussion of the higher degree result; this bound will also be found in many number theory texts.
- Waring's problem: a brief look at Chapters XX and XXI of [3] gives an introduction to the main questions.

Advanced topics for ambitious readers:

- Theta functions: there is a deep theory of theta functions, which you can read about in Chapter 10 of [6], but we will need only a few simple properties we will prove in class.
- The circle method: a beautiful but advanced introduction is given in Chapter 20 of [2].

References

[1] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley, 1999.

[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications Vol. 53, American Mathematical Society, 2004.

[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* 6th ed., also with A. Wiles, R. Heath-Brown, J. Silverman, Oxford University Press, 2008.

[4] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[5] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Lectures in Analysis Vol. 1, Princeton University Press, 2003.

[6] E. M. Stein and R. Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis Vol. 2, Princeton University Press, 2003.

[7] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, 1970.