Discrete Analogues in Harmonic Analysis

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Transcriber's Note

This is a transcription of the lectures given by professors Lillian Pierce and Eli Stein at the 2011 Princeton summer school in analysis and geometry. The lectures were delivered from July 11, 2011 to July 15, 2011 and covered the basic theory of discrete analogues in harmonic analysis.

All errors in this document are those in my transcription and interpretation of the lectures. The current version of the document is available at

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CHAPTER 1

A Quick Introduction to Harmonic Analysis in $\mathbb{R}^d$ I (E. M. Stein)

1. Prerequisites

We shall restrict our attention to the Euclidean space $\mathbb{R}^d$ with the standard metric. L^p shall denote the space of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^d} |f|^p \, dx \right)^{1/p}$$

is finite. Each L^p ($1 \leq p \leq \infty$ is a Banach space, and L^p is isometrically isomorphic to L^q, where $1 < p < \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We expect the reader to be familiar with Hölder's inequality and Minkowski's integral inequality.

We define the *Fourier transform* of $f \in L^1$ to be the integral

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx.$$

The *Fourier inversion formula* for suitable f is

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi.$$

Plancherel's theorem states that the Fourier transform is an isometry on L^2, viz.,

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

An important "multiplication operation" in harmonic analysis is *convolution*, which is defined by the integral

$$(f * g)(x) = \int f(x-y) g(y) \, dy = \int f(y) g(x-y) \, dy.$$

The Fourier transform of a convolution is the pointwise product of the Fourier transform:

$$\widehat{f * g} = \hat{f} \hat{g}.$$

2. Three continuous operators

We shall consider the following symmetries of $\mathbb{R}^d$:

- *Translations* $x \mapsto x + h$, $h \in \mathbb{R}^d$
- *Dilations* $x \mapsto \delta \cdot x$, $\delta > 0$
- *Rotations* $x \mapsto R(x)$, where R is a linear map such that $|R(x)| = |x|$.

We will exploit these special symmetries of $\mathbb{R}^d$ again and again.

Definition 1.1. The *maximal operator* of a function f is

$$\begin{aligned} M(f)(x) &= \sup_{r>0} \frac{c}{r^d} \int_{|y|\leq r} |f(x-y)|\, dy \\ &= \sup \frac{1}{m(B_x)} \int_{B(x)} |f(y)|\, dy, \end{aligned}$$

where the supremum is taken over all balls $B(x)$ centered at x.

The maximal operator came to be when G. H. Hardy wanted to study crickets. We observe that the maximal operator is invariant under all three symmetries.

Theorem 1.2. *Let* $f : \mathbb{R}^d \to \mathbb{C}$.

(a) For each $1 < p \leq \infty$, *we have*

$$\|Mf\|_{L^p} \leq A_p \|f\|_{L^p},$$

where A_p *only depends on* p *and* d.

(b) Hardy-Littlewood maximal inequality. *For each* $\alpha > 0$,

$$m\{x : (Mf)(x) > \alpha\} \leq \frac{A}{\alpha} \|f\|_{L^1},$$

where A *only depends on* d.

Note that, by definition, Mf is bigger than f. (a) shows that Mf is, in the L^p sense, not much bigger. This estimate is substituted by a *weak-type inequality* (b) in L^1 (cf. Chebychev's inequality).

We remark further that the maximal function is the "mother of all averages." If Φ is *radial*, viz.,

$$\Phi(x) = \Phi_0(|x|)$$

with Φ_0 positive, Φ_0 decreasing, and $\int \Phi\, dx = 1$, then we have

$$\left| \int f(x-y)\Phi(y)\, dy \right| \leq M(f)(x).$$

To see this, we approximate Φ by "step functions"

$$\sum c_k \frac{1}{m(B_k)} \chi_{B_k},$$

where B_k are balls centered at origin and $\sum c_k = 1$.

Proof of Theorem 1.2. Let $E_\alpha = \{x : M(f)(x) > \alpha\}$, and K a compact subset of E_α; we shall exploit the inner regularity. We pick a cover of K by fixing a ball at each point of K, and extract a finite cover $B_i = B_i(x_i)$ centered at x_i with

$$K \subseteq \bigcup_{i=1}^N B_i$$

and

$$\frac{1}{m(B_i)} \int_{B_i} |f| > \alpha.$$

We now appeal to the *Vitali covering lemma*, from which we can select pairwise-disjoint balls $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ that "almost cover" K, viz.,

$$\bigcup_j B^*_{i_j} \supseteq K,$$

where $B^*_{i_j}$ is the "three-times dilated" balls of B_{i_j}. It then follows that

$$\begin{aligned} m(K) &\leq \sum_k m(B^*_{i_j}) = 3^d \sum m(B_{i_j}) \\ &\leq \frac{3^d}{\alpha} \sum_j \int_{B_{i_j}} |f| \end{aligned}$$

□

Definition 1.3. Suppose $F \geq 0$. The *distribution function* of F is

$$\lambda(\alpha) = m\{x : F(x) > \alpha\}.$$

We note that

$$\begin{aligned} \int_0^\infty \lambda(\alpha)\, d\alpha &= \int F(x)\, dx \\ \int_0^\infty \lambda(\alpha^{1/p})\, d\alpha &= \int (F(x))^p\, dx \\ &= p \int_0^\infty \lambda(\alpha)\alpha^{p-1}\, d\alpha. \end{aligned}$$

Proof. We will strengthen

$$m\{x : (Mf)(x) > \alpha\} \leq \frac{A}{\alpha} \int_{\mathbb{R}^d} |f|\, dx$$

by

$$m\{x : (Mf)(x) > \alpha\} \leq \frac{A'}{\alpha} \int_{|f|>\alpha/2} |f|\, dx.$$

In fact, $|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$, where

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| > \frac{\alpha}{2} \\ 0 & \text{if } |f(x)| \leq \frac{\alpha}{2}. \end{cases}$$

Then we have

$$Mf \leq Mf_1 + \frac{\alpha}{2}$$

and

$$\{x : Mf > \alpha\} \subseteq \{x : Mf_1 > \frac{\alpha}{2}\},$$

so we have

$$m\{x : Mf(x) > \alpha\} \leq \frac{2A}{\alpha} \int_{|f(x)|>\alpha/2} |f|\, dx.$$

Take $F = Mf$ and $\lambda(\alpha) = m\{x : F(x) > \alpha\}$. Then

$$\begin{aligned}\int (Mf)^p &\leq A_p' \int_0^\infty \alpha^{p-1}\alpha^{-1}\left(\int_{|f|>\alpha/2} |f|\,dx\right)d\alpha\\ &= c\int |f|\left(\int_0^{2|f|}\alpha^{p-2}\,d\alpha\right)dx\\ &= c'\int |f||f|^{p-1}\,dx\\ &= c'\int |f|^p\,dx.\end{aligned}$$

□

Definition 1.4. The *Riesz potentials* of f is

$$I_\alpha(f)(x) = \frac{1}{\gamma_\alpha}\int_{\mathbb{R}^d} f(x-y)|y|^{-d+\alpha}\,dy,$$

where $0 < \alpha < d$ and

$$\gamma_\alpha = \pi^{d/2}2^\alpha\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}.$$

These are also called *fractional integrals*. We can write the above as

$$I_\alpha(f) = \int_{\mathbb{R}^d} f(x-y)K_\alpha(y)\,dy$$

where K_α is locally integrable.

Why is this operator interesting? Formally, $\widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi)$, so $I_\alpha(f) = (-\Delta)^{-\frac{\alpha}{2}}f$, where Δ is the Laplacian. (Δ is to be considered as a "basic differentiation operator of order 2", whence $(-\Delta)^{-\alpha/2}$ is the "basic differentiation operator of order α") We identify $\hat{K}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$ in the sense of distributions.

In particular, if $\alpha = 2$ and $d \geq 3$, then I_2 is the fundamental solution operator for the Laplacian Δ. When $d = 3$, I_2 is the *Newtonian potential*: note that this expression makes use of the inverse Δ^{-1} of the Laplacian.

The expression of f in terms of

$$\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d}$$

can be given as:

$$f(x) = \frac{1}{\omega}\int_{\mathbb{R}^d}\sum_{j=1}^d \frac{\partial f_j}{\partial x_j}(x-y)\frac{y_j}{|y|^d}\,dy,$$

where ω is the area of unit sphere. Therefore,

$$|f(x)| \leq cI_1(|\nabla f|)(x),$$

where ∇f is the typical gradient of f.

Theorem 1.5. *Suppose that* $1 < p < q < \infty$ *with*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}.$$

Then

$$\|I_\alpha(f)\|_{L^q} \leq A_{p,q}\|f\|_{L^p},$$

where $A_{p,q}$ depends only on d, p and q.

Observe that the fractional integral operator exhibits a dilation "relative" invariance: If $f_\delta(x) = f(\delta x)$, then

$$I_\alpha(f_\delta) = \delta^{-\alpha}(I_\alpha(f))_\delta.$$

Indeed,

$$\int f(x-y)\,\frac{dy}{(y)^{d-\alpha}}$$

is invariant, hence the factor $\delta^{-\alpha}$ comes to play. Furthermore,

$$\|f_\delta\|_{L^p} = \delta^{-\frac{d}{p}}\|f\|_{L^p}$$

It thus follows that we have the relation

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}.$$

Proof of Theorem 1.5. We claim that

$$I_\alpha(f)(x) \leq c(Mf(x))^\theta \|f\|_{L^p}^{1-\theta}$$

with $0 < \theta < 1$ and $\theta = p/q$. To see this, it suffices to consider the special case $d = 1$, $\alpha = 1/4$, $p = 2$, and $q = 4$, which then yields the inequality[1]:

$$I_1(f) \leq c(M(f))^{1/2}\|f\|_{L^2}^{1/2}.$$

Now, we split the integral in the following manner:

$$I_1(f) \quad = \quad c\int_{|y|\leq R} |y|^{-3/4} f(x-y)\,dy + c\int_{|y|>R} |y|^{-3/4} f(x-y)\,dy.$$

But

$$\begin{aligned} \int_{|y|\leq R} |y|^{-3/4} f(x-y)\,dy &\leq \left(\int_{|y|\leq R} |y|^{-3/4}\,dy\right) Mf(x) \\ &\leq cR^{1/4}M(f)(x). \end{aligned}$$

Next, by Schwarz's inequality

$$\begin{aligned} \int_{|y|>R} |y|^{-3/4} f(x-y)\,dy &\leq \left(\int_{|y|>R} |y|^{-3/2}\,dy\right)^{1/2} \|f\|_{L^2} \\ &= cR^{-1/4}\|f\|_{L^2}. \end{aligned}$$

As a result,

$$I_1(f)(x) \leq c\left\{R^{1/4}(Mf)(x) + R^{-1/4}\|f\|_{L^2}\right\}.$$

Choose

$$R = \frac{\|f\|_{L^2}^2}{(Mf(x))^2},$$

then

$$I_1(f) \leq c'm\left\{((Mf)(x))^{1/2}\,\|f\|_{L^2}^{1/2}\right\},$$

whence the desired result follows. □

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[1] the exponent 4, in the sense of conjugate exponents, is "half-way" in between the exponent 2 and the exponent ∞

Note that Theorem 1.5 fails if $p = 1$. By a duality argument, $q = \infty$ fails as well. If we replace $|y|^{-d+\alpha}$ in the definition of fractional integral by any kernel with the same "distribution" as $|y|^{-d+\alpha}$, then Theorem 1.5 still holds.

Definition 1.6 (Mihlin, Calderón, Zygmund)**.** The *singular integral* of f is

$$\begin{aligned} T(f)(x) &= \text{p. v.} \int_{\mathbb{R}^d} f(x-y)K(y)\,dy \\ &= \lim_{\varepsilon > 0} \int_{|y|>\varepsilon} f(x-y)K(y)\,dy, \end{aligned}$$

such that

- K is *homogeneous* of degree $-d$, i.e., $K_\delta = \delta^{-d}K$
- K is smooth when $x \neq 0$
- $\int_{|x|=1} K(x)\,d\sigma = 0$.

For example, consider the "forbidden kernel" $K(x) = |x|^{-d}$

Example. The *Hilbert transform* in $d = 1$ is

$$H(f)(x) = \frac{1}{\pi}\,\text{p. v.} \int_{-\infty}^{\infty} f(x-y)\,\frac{dy}{y}.$$

The principal value (p.v.) is interpreted as

$$\text{p. v.} \int f(x-y)\,\frac{dy}{y} = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} f(x-y)\,\frac{dy}{y}.$$

In a sense, the Hilbert transform is the "fundamental operator" of one-variable Fourier analysis. H is unitary on $L^2(\mathbb{R})$. Moreover, the $H^* = -H$, and $H^2 = -I$. The Hilbert transform also manifests itself in complex analysis. Indeed, the Hilbert transform maps a harmonic function to its harmonic conjugate.

Example.

$$R_{ij}(f) = \frac{\partial^2}{\partial x_i \partial x_j} I_2(f).$$

Example. The *Riesz transform* of f is defined as

$$R_j(f) = \frac{\partial}{\partial x_j} I_1(f).$$

Theorem 1.7. *If T is a singular integral operator, then T is a bounded operator L^p $(1 < p < \infty)$.*

We remark that the cases $p = 1$ and $p = \infty$ fail[2].

The method of proof is called the *Calderón-Zygmund paradigm.* Deviating from the classical method (of M. Riesz) of exploiting the power of complex analysis, we turn to the tools of real analysis. The proof has two parts: we show that the operator is bounded in L^2, and extend the result to the "L^1 theory."

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[2] Stein sayeth, "all interesting theorem should fail for $p = 1$; if a theorem holds for $p = 1$, then it is easy or semi-trivial.

PROOF OF THEOREM 1.7. We first consider the L^2 theory: we claim that

$$\widehat{(Tf)}(\xi) = m(\xi)\hat{f}(\xi).$$

where $m = \hat{K}$, taken in the senes of distributions. By Plancherel's theorem, it will then suffice to prove that

$$|m(\xi)| \leq c$$

for all ξ. If so, then we will have

$$\begin{aligned} \|T(f)\|_{L^2} &= \|\widehat{(T(f))}\|_{L^2} \\ &= \|m(\xi)\hat{f}(\xi)\|_{L^2} \\ &\leq c\|\hat{f}\|_{L^2} \\ &= c\|f\|_{L^2}. \end{aligned}$$

Let η be a radial cut-off function: η is C^∞ and $\eta(x) = 1$ if $0 < 1 \leq 1/2$, and $\eta(x) = 0$ if $|x| > 1$. Then we shall break the integral down to two parts

$$m(\xi) = \text{p.v.} \int e^{-2\pi i x\cdot\xi} K(x)\,dx = I + II$$

where

$$I = \text{p.v.} \int e^{-2\pi i x} \eta(x.r) K(x)\,dx$$

$$II = \text{p.v.} \int e^{-2\pi i x} (1 - \eta(x/r) K(x)\,dx.$$

However,

$$I = \int \left(e^{-2\pi i x\cdot\xi} - 1\right) \eta(x/r) K(x)\,dx,$$

so

$$\begin{aligned} |I| &\leq c\int_{|x|\leq r} (|x||\xi||K(x)|\,dx \\ &\leq c'|\xi| \int_{|x|\leq r} |x|^{-d+1}\,dx \\ &\leq c', \end{aligned}$$

where $r = 1/|\xi|$.

Also,

$$II = \int e^{-2\pi i x\cdot\xi}(1 - \eta(x/r))K(x)\,dx.$$

Suppose that $|\xi_j| \geq |\xi|/d^{1/2}$. Then, since

$$e^{-2\pi i x\cdot\xi} = \frac{-1}{2\pi i\xi_j} \frac{\partial}{\partial x_j}(e^{-2\pi i x\cdot\xi}),$$

it follows that

$$|II| \leq \frac{c}{|\xi_j|} \int_{|x|\geq r/2} |x|^{-d-1}\,dx \frac{c}{|\xi_j|} r^{-1} \leq c'',$$

thus establishing the easy part of the proof (via the Fourier transform).

Now we turn to the L^1 theory. We claim that

$$m\{x : |Tf(x)| > \alpha\} \leq \frac{A}{\alpha}\|f\|_{L^1}$$

for all $\alpha > 0$ (reminiscent of the weak-type estimate for the maximal function). The two main points of the proof are as follows

- *The "atoms"*[3]. Suppose $f \in L^1$ is supported in a ball B and $\int_B f\,dx = 0$. B^* be the "twice-dilated" ball of B. Then
$$\int_{^cB^*} |Tf|\,dx \leq c \int_B |f|\,dx,$$
where ${}^cB^*$ is the complement of the ball B^*. In fact, we suppose that B is centered at the origin[4]. Then
$$\begin{aligned} T(f)(x) &= \int K(x-y)f(y)\,dy \\ &= \int (K(x-y)-K(x))f(y)\,dy, \end{aligned}$$
so
$$|T(f)(x)| \leq \int_B |K(x-y)-K(x)||f(y)|\,dy.$$
Now, if $y \in B$ and $x \in {}^cB^*$ with r the radius of B, then
$$|K(x-y)-K(x)| \leq r \max_{x\in L} |\nabla K(x)|$$
where L is the line segment joining x to $x-y$. Since $|x| \geq 2r$ and $|y| \leq r$ and $|\nabla K| \leq A./|x|^{d+1}$,
$$\int_B |K(x-y)-K(x)|\,dx = \int_B |K(x-y)-K(x)|\,dx \leq cr \int_{|x|\geq 2r} |x|^{-d-1}\,dx = c'$$
if $|y| \leq r$.
- (The Calderón-Zygmund decomposition) Given $f \in L^1$ and $\alpha > 0$, we can decompose $f = g + b$, where
$$b = \sum_k b_k$$
such that
$$|g(x)| \lesssim \alpha^5 \qquad \text{and} \qquad g(x) = f(x)$$
except when $x \in E_\alpha$ with $m(E_\alpha) \lesssim \frac{1}{\alpha}\int |f|\,dx$.

  Each b_k is supported in a ball B_k, $\int b_k\,dx = 0$, and
$$\int |b_k|\,dx \lesssim \alpha m(B_k).$$
(Therefore, each b_k is an "atom") Furthermore,
$$\sum m(B_k) \lesssim \frac{1}{\alpha}\int |f|\,dx.$$
See chapters 1 and 2 of Stein, *Singular Integrals and Differentiability Properties of Functions* for details.

□

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[3] We can't actually call them *atoms*, since it is a term used in another part of harmonic analysis.

[4] for "everything is translation-invariant."

[5] This implies that $|g(x)| \leq C\alpha$ for some $C > 0$.

Here is an alternative approach to the Calderón-Zygmund decomposition that does not make use of the maximal function. We slice the Euclidean space into cubes with the mesh so large that the mean value is small:

$$\frac{1}{m(Q)}\int_Q |f|\ dx \le \alpha.$$

We subdivide each cube into 2^d pieces by bisection. If the mean value is larger than α on a subdivision, then we disregard that cube. We subdivide the "good cubes" again by bisection, and continue the process. In each step, we have

$$\alpha < \frac{1}{m(Q)}\int_Q |f| \le 2^d\alpha.$$

We have divided the space into two sets: the union of good cubes, and the union of rejected cubes. The rejected cubes are disjoint, and has the "right kind of estimate."

CHAPTER 2

Classical Discrete Operators (L. B. Pierce)

We define the convolution operator to be

$$Tf(n) = \sum_{m\in\mathbb{Z}^k} f(m)K(n-m),$$

where f has a "compact" support. Does T extend to a bounded operator on $l^p = L^p(\mathbb{Z}^k)$? Of course, l^p has the norm

$$\|f\|_{l^p} = \left(\sum_{n\in\mathbb{Z}^k} |f(n)|^p\right)^{1/p},$$

and

$$\|f\|_{l^\infty} = \sup_n |f(n)|$$

if $p = \infty$. We would like to establish

$$\|Tf\|_{l^q} \leq A_{p,q}\|f\|_{l^p}.$$

We take three simple examples:

(1) Maximal functions
(2) Fractional integral operators
(3) Hilbert transforms.

The methods for discrete operators are as follows:

(1) *Implication.*
(2) *Imitation.*
(3) *Circle method.*
(4) *Sampling.*
(5) *Method of refinements.*
(6) "Under development"

1. Maximal operators

Recall that the classical maximal operator is

$$\mathcal{M}f(x) = \sup_{r\geq 0} \frac{1}{V(r)} \int_{B_r(x)} |f(x-y)|\,dy,$$

where $V(r) = m(B_r(x))$. M maps L^p to L^p for $1 < p \leq \infty$. The discrete analogue is

$$Mf(n) = \sup_{r>0} \frac{1}{N(r)} \sum_{\substack{|m|<r\\ m\in\mathbb{Z}^k}} |f(n-m)|,$$

where $N(r) = \#\mathbb{Z}^k \cap B_r$.

Theorem 2.1. *M is a bounded operator from l^p to l^p for all $1 < p \leq \infty$, and the weak-type (1,1) estimate holds.*

PROOF OF (1) (IMPLICATION). We define the unit cube

$$Q = \{x \in \mathbb{R}^k : -\frac{1}{2} < x_j \leq \frac{1}{2}, j = 1, \ldots, k\}.$$

Then the cubes tile $\mathbb{R}^k$:

$$\mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} Q + n.$$

We shall make use of the following lemma:

Lemma 2.2. *(1) $V(r - \frac{\sqrt{k}}{2}) \leq N(r) \leq V(r + \frac{k}{2})$.*
(2) $N(r) = V(r) + O(r^{k-1})$

PROOF OF THE LEMMA. (1) is basic pythagorean theorem argument. To see that (1) implies (2), we note that $V(r) = c_k r^k$, and so

$$V\left(r \pm \frac{\sqrt{k}}{2}\right) = c_k \left(r + \frac{\sqrt{k}}{2}\right)^k.$$

$\square$

We define the companion function

$$F(x) = \sum_{n \in \mathbb{Z}^k} f(n) \chi_Q(x - n).$$

We note that $\|F\|_{L^p(\mathbb{R}^k)} = \|f\|_{l^p(\mathbb{Z}^k)}$. Indeed,

$$\int_{\mathbb{R}}^k |F|^p \, dx = \sum_{n \in \mathbb{Z}^k} \int_Q |F(x)|^p \, dx.$$

Without loss of generality, we may assume that $f \geq 0$. Then, for any $x \in Q+n$,

$$\begin{aligned} \sum_{|m|<r} f(n-m) &= \sum_{|n-m|<r} f(m) \\ &= \sum_{|n-m|<r} \int_{Q+m} F(y) \, dy \\ &\leq \int_{|n-y| \leq r + \frac{k}{2}} F(y) \, dy. \end{aligned}$$

Since

$$|x - y| \leq |x - n| + |n - y| \leq \frac{1}{2} + r + \frac{\sqrt{k}}{2} \leq r + \sqrt{k},$$

we have

$$\begin{aligned} \sum_{|m|<r} f(n-m) &\leq \int_{|n-y| \leq r + \frac{k}{2}} F(y) \, dy \\ &\leq \int_{|x-y| \leq r + \sqrt{k}} F(y) \, dy \cdot \frac{V(r + \sqrt{k})}{V(r + \sqrt{k})} \cdot \frac{1}{N(r)}. \end{aligned}$$

Now,

$$\int_{|x-y| \leq r + \sqrt{k}} F(y) \, dy \cdot \frac{1}{V(r + \sqrt{k})} \leq \mathcal{M}(F)(x)$$

and

$$\frac{V(r+\sqrt{k})}{N(r)} \leq c_k$$

for some constant c_k, so

$$Mf(n) \leq c_k \mathcal{M}F(x)$$

for each $x \in Q + n$, as was to be shown. □

Proof of (2) (imitation). The l^∞ bound is trivial. We shall show that

$$|E_\alpha| = \#\{n \in \mathbb{Z}^k : Mf(n) > \alpha\} \leq \frac{A}{\alpha}\|f\|_{l^1}.$$

Toward this end, we appeal to the Vitali covering lemma. For each $n \in E_\alpha$, there exists r_n such that

$$\sum_{|m|<r} f(n) \geq \alpha N(r_n).$$

Then

$$E_\alpha \subseteq \bigcup_{n \in E_\alpha} B_{r_n},$$

whence we can take any finite set $E \subseteq E_\alpha$ and apply the Vitali covering lemma: there exists a disjoint collection $\{B_{j_l}\}$ such that

$$\sum_{j=0}^{J} N(r_j(n_j)) \geq 3^{-k}|E|.$$

Then

$$\|f\|_{l^1} \geq \sum_{\substack{m \in \bigcup B_{r_j}(n_j) \\ n_j \in E}} \geq \alpha \sum_j N(r_j(n_j)) \geq \alpha \cdot 3^{-k}|E|,$$

The desired (1,1) inequality follows from the Marcinkiewicz interpolation theorem. □

2. Discrete fractional integration

We use a slightly different normalization for the continuous fractional operator:

$$\mathcal{I}_\lambda f(x) = \int_{\mathbb{R}^k} \frac{f(x-y)}{|y|^{k\lambda}}\, dy$$

for each $0 < \lambda < 1$.

Theorem 2.3. *For $1 < p < q < \infty$ and $0 < \lambda < 1$, we have*

$$\mathcal{I}_\lambda : L^p(\mathbb{R}^k) \to L^q(\mathbb{R}^k)$$

if and only if

$$\frac{1}{q} = \frac{1}{p} - (1-\lambda).$$

The is call the homogeneity condition.

The discrete analogue is

$$I_\lambda f(n) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{f(n-m)}{|m|^{k\lambda}}$$

for each $0 < \lambda < 1$. Note that we have left out the singular point, which we do for discrete anlogue in general.

Theorem 2.4. *For every* $0 < \lambda < 1$*, we have*

$$I_\lambda : l^p(\mathbb{Z}^k) \to l^q(\mathbb{Z}^k)$$

if and only if
(1) $\frac{1}{q} \leq \frac{1}{p} - (1 - \lambda)$
(2) $\frac{1}{q} < \lambda$, $\frac{1}{p} > 1 - \lambda$.

Proof. We first show (1) *with equality* via the imitation method. We shall establish the following pointwise estimate

$$I_\lambda f(n) \leq c(Mf(n))^{p/q} \|f\|_{l^p}^{1-p/q} :$$

here we assume without loss of generality that $f \geq 0$. Toward this end, we split the sum at R:

$$I_\lambda f(n) = I + II,$$

where

$$I = \sum_{0<|m|<R} \frac{f(n-m)}{|m|^{k\lambda}}$$

and

$$II = \sum_{|m| \geq R} \frac{f(n-m)}{|m|^{k\lambda}}.$$

Let us use the abbreviation

$$|m| \approx 2^{-j}R$$

for

$$2^{-j-1}R < |m| \leq 2^{-j}R.$$

We observe that

$$\begin{aligned}
I &\leq c \sum_{j=0}^{\infty} \sum_{|m| \approx 2^{-j}R} \frac{f(n-m)}{|m|^{k\lambda}} \\
&\leq c \sum_{j=0}^{\infty} (2^{-j}R)^{-k\lambda} \sum_{|m| \approx 2^{-j}R} f(n-m) \\
&\leq c \sum_{j=0}^{\infty} (2^{-j}R)^{-k\lambda} \sum_{|m| \leq 2^{-j}R} f(n-m) \frac{N(2^{-j}R)}{N(2^{-j}R)} \\
&\leq c' \left(\sum_{j=0} (2^{-j}R)^{-k\lambda+k} \right) Mf(n) \\
&\leq c'' R^{-k\lambda+k} Mf(n).
\end{aligned}$$

We apply Hölder's inequality to II to obtain

$$II \leq \left(\sum_{|m|\geq R} |f|^p \right)^{1/p} \left(\sum_{|m|\geq R} \frac{1}{|m|^{k\lambda p'}} \right)^{1/p'}.$$

We approximate the second factor by

$$\begin{aligned} c_r \left(\int_R^\infty r^{-kdp'} r^{k-1}\, dr \right)^{1/p'} &\approx R^{k(1-\lambda p')/p'} \\ &= R^{k(\frac{1}{p'}-\lambda}. \end{aligned}$$

Observe that

$$\frac{1}{p'} = 1 - \frac{1}{p} = X - \left(\frac{1}{q} + (1-\lambda) \right) = \lambda - \frac{1}{q},$$

which yields

$$\frac{1}{p'} - \lambda = -\frac{1}{q}.$$

It thus follows that

$$II \leq R^{-k/q} \|f\|_{l^p}.$$

We have thus shown that

$$\begin{aligned} I_\lambda f(n) &= I + II \\ &\leq c \left(Mf(n) R^{k(1-\lambda)} + R^{-k/q} \|f\|_{l^p} \right). \end{aligned}$$

We choose R such that

$$R^{k/q+k(1-\lambda)} = \frac{\|f\|_{l^p}}{Mf(n)}.$$

□

An alternative proof. Here we use the comparison method. We assume that $f \geq 0$ and construct the companion function F. We observe that each $n \neq m \in \mathbb{Z}^k$ and $u, v \in Q$ furnish a uniform constant C such that

$$|n-m|^{-k\lambda} \leq C|(n+u) - (m+v)|^{-k\lambda}.$$

For $x \in Q + n$, we see that

$$\begin{aligned} I_\lambda f(n) &= \sum \frac{f(n-m)}{|m|^{k\lambda}} \\ &= \sum_{\substack{m\in\mathbb{Z}^k \\ m\neq n}} \frac{f(m)}{|n-m|^{k\lambda}} \\ &\leq \sum_m \int_{Q+m} \frac{c \cdot F(y)}{|x-y|^{k\lambda}}\, dy \\ &\leq c \int_{\mathbb{R}^k} \frac{F(y)}{|x-y|^{k\lambda}}\, dy \\ &= \mathcal{I}_\lambda F(x). \end{aligned}$$

□

Remark. We have the *Nesting property*

$$l^{q_1} \subseteq l^{q_2}$$

if $q_1 < q_2$. Therfore, by showing the equality

$$\frac{1}{q} = \frac{1}{p} - (1-\lambda)$$

for one case, as we have done above, then we get the general inequality for free.

Remark. Note that if $f(0) = 1$ and $f(n) = 0$ if $n \neq 0$, then $f \in l^p$: this is the "delta function". But

$$I_\lambda f(n) = |n|^{-k\lambda}$$

implies

$$I_\lambda f \in l^q$$

if $\lambda q > 1$. Therefore, $1/q < \lambda$.

Observe that

$$\langle I_\lambda f, g\rangle = \sum_{\substack{m,n \\ m\neq n}} \frac{f(n)\bar{g}(m)}{|n-m|^{k\lambda}}.$$

Recalling that $I_\lambda : l^p \to l^q$ and $I_\lambda^* : l^{q'} \to l^{p'}$, we can show that $1/q < \lambda$ translates to $1/p > 1-\lambda$ by duality. It thus follows that I_λ is a self-adjoint operator.

We take a moment to introduce the *Riesz diagram.* If $T : L^p \to L^q$, we put a point where $(1/p, 1/q)$ lies. If we fix λ, then

$$\begin{aligned} \frac{1}{q} &\leq \frac{1}{p} - (1-\lambda) \\ y &\leq x - (1-\lambda) \end{aligned}$$

So we have the folloiwng shaded-in region: [diagram] Condition (2) of Theorem 2.3 states that

$$\frac{1}{q} < \lambda - \frac{1}{p} > 1-\lambda,$$

so we have the following square in the diagram: [diagram]

Example. We define

$$f(n) = \begin{cases} n^{-\gamma} & \text{if } n \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Think of $\gamma = \frac{k}{p} + \varepsilon$.

3. Hilbert transform

Recall that the continuous Hilbert transform is

$$\mathcal{H}f(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy,$$

which maps $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$. The discrete analogue is

$$Hf(n) = \frac{1}{\pi} \sum_{\substack{n\in\mathbb{Z} \\ m\neq 0}} \frac{f(n-m)}{m}.$$

Theorem 2.5. *For each $1 < p < \infty$, $H : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$.*

Proof. We shall make use of the bilinear form

$$B(f,g) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\neq n}} \frac{f(m)g(n)}{m-n}.$$

Boundedness of H on l^p is equivalent to showing

$$|B(f,g)| \leq A_p \|f\|_{l^p} \|g\|_{l^{p'}}$$

for $\frac{1}{p} + \frac{1}{p'} = 1$.

To this end, we consider the continuous analogue

$$\mathcal{B}(F,G) = \iint_{|x-y|\geq 1} \frac{F(y)G(x)}{x-y}\, dx\, dy.$$

The key idea is as follows: since $\mathcal{H}$ is bounded on L^p, we have

$$|\mathcal{B}(F,G)| \leq \|F\|_{L^p} \|G\|_{L^{p'}}.$$

We note the following regarding truncated operators:

Theorem 2.6. *Given an operator* $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ *such that*

$$\|Tf\|_{L^p} \leq A_p \|f\|_{L^p}$$

and that $T = f * K$ *for some kernel* K *with*

$$|K(x-y)| \leq \frac{A}{|x-y|^n},$$

we define

$$K_\varepsilon(x-y) = \begin{cases} K(x-y) & \textit{if } |x-y| \geq \varepsilon; \\ 0 & \textit{otherwise.} \end{cases}$$

If we set $T_\varepsilon f = f * K_\varepsilon$, *then*

$$\|T_\varepsilon f\|_{L^p} \leq A'_p \|f\|_{L^p},$$

where A'_p *is independent of* ε.[1]

Lemma 2.7.

$$|\mathcal{B}(F,G) - B(f,g)| = O(\|f\|_{l^p} \|g\|_{l^{p'}}).$$

Proof of the lemma. We claim that

$$\iint_{\substack{|x-y|\geq 1 \\ x\in Q+n \\ y\in Q+m}} \frac{dx\, dy}{x-y} = \begin{cases} \frac{1}{n-m} + O\left(\frac{1}{|n-m|^2}\right) & \text{if } n \neq m; \\ 0 & \text{if } n = m. \end{cases}$$

If we take the claim for granted, then

$$\begin{aligned} &|\mathcal{B}(F,G) - B(f,g)| \\ \leq\ & c \left| \sum_{n,m} \iint_{\substack{|x-y|\geq 1 \\ x\in Q+n \\ y\in Q+m}} \frac{F(y)G(x)}{x-y}\, dy\, dy - \sum_{\substack{n,m \\ n\neq m}} \frac{f(m)g(n)}{n-m} \right|. \end{aligned}$$

---

[1] See Stein, *Harmonic Analysis* for details.

By the construction of our companion functions, we have $F(y) = f(m)$ and $G(x) = g(n)$ in the above integral, and so

$$\begin{aligned}
|\mathcal{B}(F,G) - B(f,g)| &\leq c\left|\sum_{n,m}\iint_{\substack{|x-y|\geq 1\\ x\in Q+n\\ y\in Q+m}}\frac{F(y)G(x)}{x-y}\,dy\,dy - \sum_{\substack{n,m\\ n\neq m}}\frac{f(m)g(n)}{n-m}\right| \\
&\leq \sum_{n,m}|f(m)g(n)|\left|\iint_{\substack{|x-y|\geq 1\\ x\in Q+n\\ y\in Q+m}}\frac{dy\,dx}{x-y} - \frac{1}{n-m}\right| \\
&\leq 2c\sum_{n,m}\frac{|f(m)g(n)|}{1+|n-m|^2},
\end{aligned}$$

where the last inequality follows from the claim. By Young's inequality, we have

$$c\sum_{n,m}\frac{|f(m)g(n)|}{1+|n-m|^2} \leq \left\|\frac{1}{1+x^2}\right\|_{L^1}\|fg\|_{l^1},$$

and by Hölder's inequality we have

$$\left\|\frac{1}{1+x^2}\right\|_{L^1}\|fg\|_{l^1} \leq c\|f\|_{l^p}\|g\|_{l^{p'}}.$$

□

We now proceed to verify the claim. If $n = m$, then the integral is 0. If $n \neq m$ and $|n-m| \geq 2$, then

$$\left|\int_{m-1/2}^{m+1/2}\int_{n-1/2}^{n+1/2}\frac{dx\,dy}{x-y} - \frac{1}{n-m}\right| \leq \int_{m-1/2}^{m+1/2}\int_{n-1/2}^{n+1/2}\left|\frac{1}{x-y} - \frac{1}{n-m}\right|dx\,dy.$$

By the regrouping

$$\frac{n-m-(x-y)}{(x-y)(n-m)},$$

we have the estimate

$$\begin{aligned}
\int_{m-1/2}^{m+1/2}\int_{n-1/2}^{n+1/2}\left|\frac{1}{x-y} - \frac{1}{n-m}\right|dx\,dy &\leq \int_{m-1/2}^{m+1/2}\int_{n-1/2}^{n+1/2}\frac{|x-n|+|y-m|}{|x-y||n-m|} \\
&= O\left(\frac{1}{|n-m|^2}\right),
\end{aligned}$$

for $|x-n| \leq 1$ and $|y-m| \leq 1$. This completes the proof. □

CHAPTER 3

A Quick Introduction to Harmonic Analysis in $\mathbb{R}^d$ II (E. M. Stein)

1. The Role of Curvature

Definition 3.1. A *(smooth) curve* in $\mathbb{R}^d$ is a C^∞-function $\gamma : [-1,1] \to \mathbb{R}^d$ with $\gamma(0) = 0$ and $|\gamma'(t)| > 0$ for all $t \in [-1,1]$.

We can consider operators on a curve γ: for example, the Hilbert transform

$$H_\gamma(f)(x) = \mathrm{p.\,v.} \int_{-1}^{1} f(x - \gamma(t))\, \frac{dt}{t}$$

and the maximal function.

$$M_\gamma(f)(x) = \sup_{0<h\leq 1} \frac{1}{2h} \int_{|t|\leq h} |f(x-\gamma(t))|\, dt.$$

Is there an $L^p(\mathbb{R}^d)$ theory for H_γ and M_γ?

(1) Yes, if γ is a straight-line segment;
(2) No, in general, but:
(3) Yes, if γ has some "curvature" (near $t = 0$).

Why is there a good theory for straight lines? We can always rotate the line to be axis-parallel, whence the case is reduced to the theory on $\mathbb{R}$. Note, however, that "local flatness" is decidedly not enough: we can construct a curve which is "very flat" at every given point which nevertheless does not admit a good theory for any p.

What is a good model of non-flat curve with an L^p-theory? The more precise version of hypothesis (3) can be stated as follows: For some $N \in \mathbb{N}$, the vectors

$$\gamma'(0), \gamma''(0), \cdots, \gamma^{(n)}(0)$$

span $\mathbb{R}^d$.

Example. Here is a typical example: $\gamma(t) = (t, t^2)$, the parabola in $\mathbb{R}^2$. Note that $\gamma'(0) = (1,0)$ and $\gamma''(0) = (0,2)$, whence the hypothesis is satisfied.

Theorem 3.2. *Under the curvature hypothesis (3) above, both H_γ and M_γ are bounded operators on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.*

The methods of proving boundedness results for the standard maximal function, fractional integral, and singular integrals fail. In particular, no results are known for $p = 1$ for H_γ and M_γ: we do not know, for example, if the weak-type estimates hold.

The key tools here are *oscillatory integrals*. Here is the one-dimensional example; here I is the standard example, and J a variant:

$$\begin{aligned} I(\lambda) &= \int_a^b e^{i\lambda\Phi(x)}\,dx. \\ J(\lambda) &= \int_a^b e^{i\lambda\Phi(x)}\psi(x)\,dx \end{aligned}$$

Φ is the *phase*, Φ is *real-valued*, ψ is the *amplitude*. We would like to know how the integrals behave as λ tends to infinity. In particular, we would like to know the kind of estimates we can make that are independent of the integral.

Here is an outline of the idea of the proof of Theorem 3.2 in the case when γ is the parabola in $\mathbb{R}^2$ and $p = 2$: (i) oscillatory integrals, (ii) boundedness of H_γ, (iii) boundedness of M_γ.

Theorem 3.3. *Let*

$$I(\gamma) = \int_a^b e^{i\lambda\Phi(x)}\,dx,$$

where Φ is of class C^2. If $|\Phi'(x)| \geq 1$ and Φ' monotonic, then $|I(\lambda)| \leq c|\lambda|^{-1}$. Alternatively, if $|\Phi''(x)| \geq 1$, then $|I(\lambda)| \leq c|\lambda|^{-1/2}$ as $|\lambda| \to \infty$. Here the bound c is independent of a and b

If $\Phi(x) = x$, then

$$\int_a^b e^{i\lambda x}\,dx = \frac{1}{i\lambda},$$

whence the decay is clearly of order $|\lambda|^{-1}$. Examine also $\Phi(x) = x^2$.

PROOF. We first write

$$e^{i\lambda\Phi(x)} = \frac{1}{i\lambda\Phi'(x)}\frac{d}{dx}\left(e^{i\lambda\Phi(x)}\right),$$

so that

$$I(\lambda) = \frac{1}{i\lambda}\int_a^b \frac{1}{\Phi'(x)}\frac{d}{dx}\left(e^{i\lambda\Phi(x)}\right)\,dx,$$

whence we can integrate by parts. We now take the absolute value of each term. The $1/\Phi'(x)$ looks bad, but fear not—$\Phi'(x)$ is monotonic, so we can remove the absolute value. The estimate now follows from simple computation.

In the second alternative, we shall write

$$I(\lambda) = I + II,$$

such that

$$\begin{aligned} I &= \int_{c-\lambda}^{c+\lambda} e^{i\lambda\Phi}\,dx \\ II &= \text{the complementary part of I}, \end{aligned}$$

where c is the unique point where $\Phi'(c) = 0$. By doing so, we have singled out the "bad point." We first make the trivial estimate

$$|I| \leq 2\delta,$$

which will turn out to be good enough—we shall make δ very small. Moreover, we have

$$|II| \leq \frac{2}{|\lambda|^{\delta}}.$$

We still haven't picked δ yet: the right choice of δ is

$$\delta = |\lambda|^{-1/2},$$

which yields the desired estimate. □

Corollary 3.4. *Let*

$$J(\lambda) = \int_a^b e^{i\lambda\Phi(x)}\psi(x)\,dx.$$

If $|\Phi'(x)| \geq 1$ *and* Φ' *monotonic, then*

$$|J(\lambda)| \leq \frac{c}{|\lambda|} \cdot \left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

If, instead, $|\Phi'(x)| \geq 1$*, then*

$$|J(\lambda)| \leq \frac{c}{|\lambda|}^{1/2} \left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

Proof. We write

$$\begin{aligned} I^u(\lambda) &= \int_a^u e^{i\lambda\Phi(x)}\,dx \\ J(\lambda) &= \int_a^b \frac{d}{du}(I^u(\lambda))\psi(u)\,du \end{aligned}$$

and integrate by parts. □

We now examine the Hilbert transform on the parabola $\gamma = (t, t^2) \in \mathbb{R}^2$:

$$H_\gamma(f) = \text{p. v.} \int_{-\infty}^{\infty} f(x-t, y-t^2)\,\frac{dt}{t}.$$

This is a convolution-type integral, and so we are interested in the Fourier transform of the above integral: indeed, we have

$$\widehat{H_\gamma(f)} = m(\xi,\eta)\hat{f}(\xi,\eta),$$

where

$$m(\xi,\eta) = \text{p. v.} \int_{-\infty}^{\infty} e^{-2\pi i(\xi t + \eta t^2)}\,\frac{dt}{t}.$$

Theorem 3.5. *Let*

$$m(\xi,\eta) = \text{p. v.} \int_{-\infty}^{\infty} e^{-2\pi i(\xi t + \eta t^2)}\,\frac{dt}{t}$$

Then $|m(\xi,\eta)| \leq A$ *for all* (ξ,η).

Corollary 3.6. *We can write*

$$H_\gamma(f) = \text{p. v.} \int_{-\infty}^{\infty} f(x-t, y-t^2)\,\frac{dt}{t},$$

and $f \mapsto H_\gamma(f)$ *is bounded on* $L^2(\mathbb{R}^2)$.

Proof. To this end, we consider new “dilations”

$$\delta \circ (\xi, \eta) = (\delta \xi, \delta^2 \eta)$$

for each $\delta > 0$. We also consider a new “norm”

$$\rho(\xi, \eta) = |\xi| + |\eta|^{1/2}.$$

These are the natural notions of dilation and norm for the parabola; indeed:

$$\rho(\delta \circ (\xi, \eta)) = \delta \rho(\xi, \eta).$$

We decompose the integral into an infinite sum of diadic parts:

$$\text{p. v.} \int_{-\infty}^{\infty} e^{-2\pi i(\xi t + \eta t^2)} \, \frac{dt}{t} = \sum_{-\infty < k < \infty} \int_{2^k}^{2^{k+1}} \int_{-\infty}^{\infty} e^{-2\pi i(\xi t + \eta t^2)} \, \frac{dt}{t}.$$

Note that

$$\int_{2^k}^{2^{k+1}} e^{-2\pi i(\xi t + \eta t^2)} \, \frac{dt}{t} = \int_1^2 e^{-2\pi i(2^k \xi t + 2^{2k} \eta t^2)} \, \frac{dt}{t}$$

by introducing a proper dilation: here each integral is taken over half-open interval $[a, b)$. We then have

$$m_0(2^k \xi, 2^{2k} \eta),$$

where

$$m_0(\xi, \eta) = \int_1^2 e^{-2\pi i(\xi t + \eta t^2)} \, \frac{dt}{t}.$$

We shall write

$$m_0(\xi, \eta) = \int_1^2 e^{i \lambda \Phi(t)} \psi(t) \, dt.$$

Claim. *(1)* $|m_0(\xi, \eta)| \leq A$ *for all* (ξ, η)
(2) $m_0(\xi, \eta)$ *is a smooth function with* $m_0(0,0) = 0$*; therefore,* $|m_0(\xi, \eta)| \leq c\rho(\xi, \eta)$.
(3) $|m_0(\xi, \eta)| \leq c(\rho(\xi, \eta))^{-1/2}$.

The assertions (1) and (2) are easy. To show (3), we write

$$\begin{aligned} m_0(\xi, \eta) &= \int_1^2 e^{i\lambda\Phi(t)} \psi(t) \, dt \\ &= \int_1^2 e^{-2\pi i(\xi t + \eta t^2)} \, \frac{dt}{t}. \end{aligned}$$

We now make an estimate using the first and the second derivative tests. We first consider the case when ξ is very large, say, $|\xi| \geq 10|\eta|$. Then

$$\Phi(t) = -2\pi \left(t + \frac{\eta}{\xi} \cdot t^2 \right),$$

$\lambda = \xi$ and $\psi(t) = \frac{1}{t}$. Therefore, $|\Phi'(t)| \geq 1$ and

$$|m_0(\xi, \eta)| \leq \frac{c}{|\xi|} \leq \frac{c}{\rho(\xi, \eta)^{1/2}}$$

if $|\xi| \geq 10|\eta|$, and so $\rho(\xi, \eta) \geq 1$.

If $|\xi| \leq 10|\eta|$, then we write

$$\Phi(t) = -2\pi \left(t\frac{\xi}{\eta} + t^2 \right),$$

so that $\lambda = \eta$. Then $|\Phi'(t)| \geq 1$ and

$$m_0(\xi, \eta) \leq c\eta^{-1/2} \leq c\rho(\xi, \eta)^{-1/2}.$$

Now,

$$\begin{aligned} m(\xi, \eta) &= \sum_k m_0(2^k\xi, 2^{2k}\eta) \\ &= \sum_{2^k\rho(\xi,\eta)\leq 1} m_0(2^k\xi, 2^{2k}\eta) + \sum_{2^k\rho(\xi,\eta)>1} m_0(2^k\xi, 2^{2k}\eta). \end{aligned}$$

The first sum is estimated as follows:

$$\begin{aligned} c\sum \rho(2^k\xi, 2^{2k}\eta) &= c\left(\sum_{2^k\rho(\xi,\eta)\leq 1} 2^k\right) \\ &\leq c'. \end{aligned}$$

The second sum is estimated as follows:

$$\begin{aligned} \sum_{2^k\rho(\xi,\eta)>1} \rho(2^k\xi, 2^{2k}\eta)^{-1/2} &= \left(\sum_{2^k\rho(\xi,\eta)>1} 2^{-k/2}\right)(\rho(\xi,\eta))^{-1/2} \\ &\leq c'. \end{aligned}$$

□

We mention some other results involving oscillatory integrals. Suppose

$$P(t) = a_0 + a_1 t + \cdots + a_k t^k$$

is a polynomial with real coefficients. Then Corollary 3.6 implies that

$$\left| \text{p. v.} \int_{-\infty}^{\infty} e^{iP(t)} \frac{dt}{t} \right| \leq M,$$

where M is independent of $a_0, a_1, \ldots, a_k$.

Corollary 3.7. *We suppose that* $Q(t) = (Q_1(t), \ldots, Q_d(t))$ *is a polynomial function from* $\mathbb{R}$ *to* $\mathbb{R}^d$. *Then the "Hilbert transform"*

$$H_Q(f)(x) = \text{p. v.} \int_{-\infty}^{\infty} f(x - Q(t)) \frac{dt}{t}$$

is bounded on $L^2(\mathbb{R}^d)$.

We now turn to the boundedness of M_γ, where γ is still fixed to be the parabola. To this end, we consider *non-isotropic dilations* in $\mathbb{R}^d$:

$$x \mapsto \delta \circ x = (\delta^{a_1}x_1, \delta^{a_2}x_2, \ldots, \delta^{a_d}x_d).$$

Here $a_1, \ldots, a_d$ are fixed strictly-positive exponents. The quantity $a_1 + a_2 + \cdots + a_d = Q$ is the *homogeneous dimension* of $\mathbb{R}^d$. With this definition, we have:

$$d(\delta \circ x) = \delta^Q \, dx.$$

In general, many of the basic results, such as the above theorems, have valid extensions where the isotropic dilations

$$x \mapsto \delta x = (\delta x_1, \delta x_2, \ldots, \delta x_d)$$

are replaced by non-isotropic dilations: of course, the extensions will have to be formulated properly.

Example. Consider the maximal function $\tilde{M}$

$$\tilde{M}(f)(x) = \sup_{\delta>0} \frac{1}{m(B_\delta)} \int_{B_\delta} |f(x-y)|\, dy.$$

where

$$B_\delta = \delta \circ B = \{x = \delta \circ y, y \in B = \text{ unit ball } \}$$

and

$$m(B_\delta) = \delta^Q m(B) = c\delta^Q.$$

While this is quite a different maximal operator, the boundedness proof essentially follows by simply imitating the proof for the isotropic dilation case. Indeed:

Proposition 3.8. *(a)* $\|\tilde{M}(f)\|_{L^p} \leq A\|f\|_{L^p}$ $(1 < p \leq \infty)$
(b) $m\{x : \tilde{M}(f) > \alpha\} \leq \frac{A}{\alpha}\|f\|_{L^1}$ *for all* $\alpha > 0$.

This follows from the non-isotropic Vitali lema: there is a $c > 0$ such that if $B_{\delta_1}(x_1)$ and $B_{\delta_2}(x_2)$ intersect and $\delta_1 \geq \delta_2$, then

$$B_{c\delta_1}(x_1) \supseteq B_{\delta_2}(x_2).$$

Theorem 3.9. *Let*

$$M_\gamma(f)(x) = \sup_{0<r} \frac{1}{2r} \int_{|t|\leq r} |f(x_1 - t, x_2 - t^2)|\, dt.$$

Then

$$\|M_\gamma(f)\|_{L^2} \leq A\|f\|_{L^2}.$$

Proof. It suffices to assume that $f \geq 0$, and establish

$$\left\|\sup_{k\in\mathbb{Z}} |A_k(f)|\right\|_{L^2} \leq C\|f\|_{L^2},$$

where

$$A_k f = 2^{-(k+1)} \int_{|t|\leq 2^k} f(x_1 - t, x_2 - t^2)\, dt.$$

These are very "singular averages."

Let φ be a smooth bump function, viz., a C^∞ function on $\mathbb{R}^2$ such that $\varphi \geq 0$, φ supported in $|x| \leq 1$, and

$$\int \varphi(x)\, dx = 1.$$

Setting

$$\varphi_k(x) = 2^{-3k}\varphi(2^{-k}x_1, 2^{-2k}x_2),$$

we see that $\int \varphi_k(x)\, dx = 1$ and φ_k is supported on B_{2^k} with $m(B_{2^k}) = 2^{3k}m(B)$.

We write

$$M_k(f) = f * \varphi_k = \int_{\mathbb{R}^2} f(x-y)\varphi_k(y)\, dy.$$

so that

$$\begin{aligned} M_k(f)(x) &\leq c\frac{1}{m(B_{2^k})} \int_{B_{2^k}} f(x-y)\, dy \\ &\leq c\tilde{M}(f). \end{aligned}$$

It then follows that

$$\left\|\sup_k |M_k(f)|\right\|_{L^2} \leq c\|f\|_{L^2},$$

whence it suffices to compare M_k with A_k.

This coparison is done by a square function $\mathcal{S}$ defined by

$$(\mathcal{S}(f)(x))^2 = \sum_{k\in\mathbb{Z}} (A_k(f)(x) - M_k f(x))^2.$$

Observe that

$$|A_k(f)(x) - M_k(f)(x)| \leq \mathcal{S}(f)(x),$$

and so

$$\sup_k A_k(f)(x) \leq \mathcal{S}(f)(x) + c\tilde{M}(f)(x).$$

It thus suffices to prove

Lemma 3.10.

$$\|\mathcal{S}(f)\|_{L^2} \leq c\|f\|_{L^2}.$$

Proof of the lemma. Let $\hat{\varphi}(\xi)$ be the Fourier transform of φ, which is

$$\int_{\mathbb{R}^2}\int e^{-2\pi i(\xi\lambda_1+\eta x_2)}\varphi(x_1,x_2)\,dx.$$

We note that

1. $\hat{\varphi}$ is smooth.
2. $\hat{\varphi}(0) = 1$
3. $\hat{\varphi}$ is rapidly decreasing at infinity, so in particular $|\hat{\phi}(\xi,\eta)| \leq c(\rho(\xi,\eta))^{-1/2}$
4. $\hat{\phi}_k(\xi,\eta) = \hat{\phi}(2^k\xi, 2^{2k}\eta)$.

Therefore,

$$(\widehat{A_k - M_k})(f)(\xi,\eta) = m_k(\xi,\eta) - \hat{\varphi}_k(\xi,\eta),$$

where

$$m_k(\xi,\eta) = 2^{-k-1}\int_{|t|\leq 2^k} e^{-2\pi i(\xi t+\eta t^2)}\;dt = m_0\left(2^k\xi, 2^{2k}\eta\right).$$

We note that

1. m_0 is smooth.
2. $m_0(0,0) = 1$
3. $|m_0(\xi,\eta)| \leq c\rho(\xi,\eta)^{-t/2}$.

We thus conclude that

$$(\widehat{A_k - M_k})(f) = \Delta_k\cdot\hat{f},$$

where

$$\Delta_k(\xi,\eta) = m_k(\xi,\eta) - \hat{\varphi}_k(\xi,\eta) = \Delta_0(2^k\xi, 2^{2k}\eta)$$

and

$$\begin{aligned}|\Delta_0(\xi,\eta)| &\leq c(|\xi|+|\eta|) \leq c\rho(\xi,\eta) \text{ if } \rho\leq 1;\\ |\Delta_0(\xi,\eta)| &\leq c(\rho(\xi,\eta))^{-1/2} \text{ if } \rho\geq 1.\end{aligned}$$

But by Plancherel's theorem,

$$\|(A_k - M_k)(f)\|_{L^2}^2 = \int_{\mathbb{R}^2} |\Delta_k(\xi,\eta)|^2|\hat{f}(\xi,\eta)|^2\,d\xi\,d\eta,$$

so

$$\begin{aligned}
\|S(f)\|_{L^2}^2 &= \sum_k \|(A_k - M_k)(f)\|_{L^2}^2 \\
&= \int \left(\sum_k |\Delta_k(\xi,\eta)|^2 \right) |\hat{f}|^2 \, d\xi \, d\eta \\
&\leq c^2 \int |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta \\
&= c^2 \int |f(x)|^2 \, dx,
\end{aligned}$$

for $\sum_k |\Delta_k(\xi)|^2 \leq c^2$. ☐

The proof now follows from the lemma. We also note that

$$\begin{aligned}
\sum_{2^k \rho \leq 1} 2^{2k} \rho(\xi,\eta)^2 &\leq c_1; \\
\sum_{2^k \rho \geq 1} 2^{-k} \rho(\xi,\eta)^{-1} &\leq c_1.
\end{aligned}$$

☐

Further readings in Stein, *Harmonic Analysis*:

- Oscillatory integrals: Chapter 8, sections 1-3
- Maximal functions and singular integrals on curved varieties: Chapter 9, section 1.2 and 2.

CHAPTER 4

Discrete Hilbert Transform on the Parabola (L. B. Pierce)

In this section, we shall prove the L^2-boundedness of the discrete Hilbert transform on the parabola.

1. Preliminaries

We shall start off by considering a few general properties of discrete operators.

Proposition 4.1. *If* $Tf(n) = \sum_m f(m)K(n,m)$ *and* T *is translation-invariant, viz.,* $T \circ \tau_h = \tau_h \circ T$, *then there exists a function* f_n *such that* $K(n,m) = K_0(n-m)$.

Example. Let

$$Tf(n) = \sum_{m \in \mathbb{Z}^k} f(n - P(m))K(m),$$

where $P : \mathbb{Z}^k \to \mathbb{Z}^l$ is a "polynomial." This is easily seen to be a translation-invariant operator.

Example. We let

$$Tf(n,t) = \sum_{m \in \mathbb{Z}^k} f(n - m, t - n \cdot m)K(m),$$

where $n \in \mathbb{Z}^k$ and $t \in \mathbb{Z}$. This is *not* a translation-invariant operator, but it *is* translation-invariant with respect to t. We call such a T a *quasi-translation-invariant* operator.

Proposition 4.2. *If* T *is translation-invariant, then* T *is bounded on* l^2 *if and only if there exists an* $m \in L^\infty(\mathbb{T}^k)$ *such that*

$$\tilde{T} = \mathscr{F} T \mathscr{F}^{-1}$$

is of the form

$$\tilde{T}f(x) = m(x)f(x).$$

In this case, we have

$$\|m(x)\|_{L^\infty} = \|T\|_{l^2 \to l^2}.$$

Here $\mathscr{F}$ is the *Fourier transform* $f \mapsto \hat{f}$ from $\mathbb{Z}^k$ to $\mathbb{T}^k$, defined by

$$f(\theta) = \sum_{n \in \mathbb{Z}^k} f(n) e^{-2\pi i n \cdot \theta},$$

and $\mathscr{F}^{-1}$ the *inverse Fourier transform* $h \mapsto h^\vee$ from $\mathbb{T}^k$ to $\mathbb{Z}^k$ by

$$h^\vee(n) = \int_0^1 h(\theta) e^{2\pi i n \cdot \theta} \, d\theta.$$

Instead of considering $\widehat{(Tf)}(\theta)$, we can consider $\hat{f}(\theta)$ via the identity

$$\widehat{(Tf)}(\theta) = m(\theta)\hat{f}(\theta),$$

where m is the *Fourier multiplier.*

Proposition 4.3. *If T is translation-invariant and bounded on l^p for some $1 < p < \infty$, then T is bounded on l^2 and*

$$\|T\|_{l^p \to l^p} = \|T\|_{l^2 \to l^2}.$$

We observe that

$$\widehat{(Tf)}(\theta) = \sum_n \sum_m f(n - P(m))K(m)e^{-2\pi i n \cdot \theta}.$$

By switching the order of summation and sending $n \mapsto P(m)$, we have

$$\begin{aligned} \widehat{(Tf)}(\theta) &= \sum_n \sum_m f(n - P(m))K(m)e^{-2\pi i n \cdot \theta} \\ &= m(\theta)\hat{f}(\theta), \end{aligned}$$

whence

$$m(\theta) = \sum_m e^{-2\pi i P(m)\theta} K(m).$$

Today's main thereom is the following:

Theorem 4.4. *The* discrete Hilbert transform on the parabola

$$H_{par} f(n) = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{f(n_1 - m, n_2 - m^2)}{m},$$

then H_{par} a bounded operator on l^2.

From the above discussion, we see that it suffices to show

$$\|m\|_{L^\infty[0,1]} \leq c.$$

To this end, we write the Fourier multiplier as

$$m(\theta, \varphi) = \sum_{m \neq 0} \frac{e^{-2\pi i (m^2\theta + m\varphi)}}{m}.$$

We need to show that

$$\left| \sum_{m \neq 0} \frac{e^{-2\pi i (m^2\theta + m\varphi)}}{m} \right| \leq A$$

independent of θ and φ.

Note that we cannot pass the absolute value inside, for the result is a harmonic series. We would like to avoid integer phases, thereby avoiding

$$e^{2\pi i k} = 1.$$

Indeed, anytime we have $(\theta, \varphi) \approx (\frac{a}{q}, \frac{b}{q})$, we have to "worry" about the distribution of (m^2, m) modulo q. As q gets large, we get to "worry less." Our goal is to make these "worries" precise.

If $(\theta, \varphi) \approx (\frac{a}{q}, \frac{b}{q})$ with a "small" q, then we shall say that (θ, φ) lie in a *major arc*. If $(\theta, \varphi) \approx (\frac{a}{q}, \frac{b}{q})$ with a "large" q, then we shall say that (θ, φ) lie in a *minor* *arc*. We will also define these terms precisely, including the symbol $\approx$. To this end, we shall make use of three tools from number theory: Diophantine approximations, exponential sums and Weyl bounds, and theta functions.

2. Diophantine approximation

Theorem 4.5 (Dirichlet's approximation principle)**.** *If $\theta \in [0,1]$ and $N \geq 1$, then there exist integers $1 \leq a \leq q \leq N$ with $(a,q) = 1$ such that*

$$\left|\theta - \frac{a}{q}\right| \leq \frac{1}{qN},$$

where $|\cdot|$ is the "distance to $\mathbb{Z}$."

Sketch of proof. We look at the $N+1$ numbers $0\cdot\theta, 1\cdot\theta, \ldots, N\cdot\theta$ modulo 1. If we divide $[0,1]$ into N intervals

$$[0,1/N], [1/N, 2/N], \cdots, [(N-1)/N, 1],$$

then by the pigeonhole principle there are $1 \leq a, b \leq N$ such that

$$|a\theta - b\theta| \leq \frac{1}{N}$$

modulo 1. Then we can find $r \in \mathbb{Z}$ such that

$$|a\theta - b\theta - r| \leq \frac{1}{N},$$

whence the desired result follows. ☐

3. Exponential sums

We shall consider three types of exponential sums. A *linear exponential sum* is

$$S = \sum_{n=1}^{N} e^{2\pi i n\theta},$$

which has the geometric sum

$$\begin{aligned} S &= \sum_{n=1}^{N} e^{2\pi i n\theta} \\ &= \frac{e^{2\pi i\theta}(1 - e^{2\pi i N\theta})}{1 - e^{2\pi i\theta}} \\ &= \frac{e^{2\pi i\theta} e^{\pi i N\theta}(e^{-\pi i N\theta} - e^{\pi i N\theta})}{e^{\pi i\theta}(e^{-\pi i\theta} - e^{\pi i\theta})} \\ &= e^{\pi i(N+2)\theta}\left(\frac{\sin(\pi N\theta)}{\sin(\pi\theta)}\right). \end{aligned}$$

We have the trivial bound

$$|S| \leq N$$

everywhere—especially near zero—and so

$$|S| \leq \min\left(N, \frac{1}{|\sin \pi\theta|}\right).$$

We now observe that

$$|\sin \pi\alpha| \geq \frac{1}{2}\|\alpha\|,$$

where $\|\alpha\|$ is the distance from α to the nearest integer. It follows that we have the bound

$$|S| \leq c \min\left(N, \frac{1}{\|\theta\|}\right).$$

There are two types of *quadratic sums.* A *Gauss sum* $S(a,q) = S(a/q)$ for any a and q with $(a,q)=1$ is

$$S(a/q) = \sum_{n=1}^{q} e^{2\pi i a n^2/q}.$$

A *Weyl sum*, which subsumes Gauss sums, is defined as

$$S = \sum_{n=1}^{N} e^{2\pi i P(n)\theta},$$

where $P(n) : \mathbb{Z} \to \mathbb{Z}$ is a polynomial and $\theta \in [0,1]$ a real number.

The Gauss sum is easy to characterize if we are summing over rationals. Indeed, if $\theta = a/q$, then

$$\sum_{n=1}^{q} e^{2\pi i n a/q} = \begin{cases} q & \text{if } a \equiv 0 \pmod{q}; \\ 0 & \text{if } a \not\equiv 0 \pmod{q}. \end{cases}$$

This follows from the orthogonality of characters

$$\chi^{(a)}(n) = e^{2\pi i a n}.$$

Theorem 4.6 (Square-root cancellation for Gauss sums)**.** $|S(a/q)| \leq c q^{1/2}$.

PROOF. We make use of the so-called "squaring trick":

$$\begin{aligned} |S(a/q)|^2 &= S(a/q)\overline{S(a/q)} \\ &= \sum_n \sum_m e^{2\pi i (m^2 - n^2)a/q}. \end{aligned}$$

Substituting $m \mapsto n + l$ yields

$$\begin{aligned} |S(a/q)|^2 &= \sum_n \sum_m e^{2\pi i (m^2 - n^2)a/q} \\ &= \sum_l \left(\sum_n e^{2\pi i (2nla/q)} \right) e^{2\pi i l^2 a/q}, \end{aligned}$$

where

$$\sum_n e^{2\pi i (2nla/q)}$$

is linear in n. We apply the special case computed above by considering three cases.

If q is odd, then

$$|S(a/q)|^2 = q.$$

If q is even, we consider two subcases. If $q \equiv 2 \pmod{4}$, then $2l \equiv 0 \pmod{4}$, and so

$$|S(a/q)|^2 = q\left(e^{2\pi i q a} + e^{\pi i q a/2}\right) = q(1-1) = 0.$$

If $q \equiv 0 \pmod{4}$, then $2l \equiv 0 \pmod{4}$, and so

$$|S(a/q)|^2 = q(e^{2\pi i q a} + e^{\pi i q a/2}) = q(1+1) = 2q.$$

It thus follows that

$$|S(a/q)| \leq c\sqrt{q},$$

as was to be shown. □

We now turn to the general Weyl sum of degree two. H. Weyl wanted to know whether the sequence $(n\theta)_{n=1}^{\infty}$ modulo 1, where θ is an irrational number, is *equidistributed*, viz.,

$$\lim_{N\to\infty} \#\left\{1 \le n \le N : n\theta \pmod{1} \in [a,b]\right\} = b - a.$$

The answer is yes, and we have the more general *Weyl's criterion*: namely, the sequence $(a_n)_{n=1}^{\infty}$ is equidistributed on $[0,1]$ if, for all $l \neq 0$,

$$\sum_{n=1}^{N} e^{2\pi i a_n l} = o(N)$$

as $N \to \infty$. This line of thought led to the following theorem:

Theorem 4.7 (Weyl bound, degree 2)**.** *Let $\theta \in \mathbb{R}$ and suppose that $(a,q) = 1$ such that*

$$\left|\theta - \frac{a}{q}\right| = \frac{1}{q^2}.$$

Then, for

$$S(\theta,\varphi) = \sum_{n=1}^{N} e^{2\pi i(n^2\theta + n\varphi)},$$

the following bound holds:

$$|S(\theta,\varphi)| \le c\left(\frac{N}{q^{1/2}} + q^{1/2}\right)(\log q)^{1/2}.$$

Proof. We appeal to the squaring trick again, which is also called the *Weyl differencing process*:

$$\begin{aligned}|S(\theta,\varphi)|^2 &= S(\theta,\varphi)\overline{S(\theta,\varphi)}\\ &= \sum_{1\le m,n\le N} e^{2\pi i}\left((m^2-n^2)\theta + (m-n)\varphi\right).\end{aligned}$$

The substitution $m = n + l$ yields

$$\begin{aligned}|S(\theta,\varphi)|^2 &= \sum_{1\le m,n\le N} e^{2\pi i}\left((m^2-n^2)\theta + (m-n)\varphi\right)\\ &= \sum_{\substack{1\le n\le N\\ 1-n\le l\le N-n}} e^{2\pi i((l^2+2nl)\theta + l\varphi)}\\ &= \sum_{|l|<N} e^{2\pi i(l^2\theta + l\varphi)} \sum_{1\le n,n+l\le N} e^{2\pi i(2nl\theta)}.\end{aligned}$$

Since

$$\left|\sum_{1\le n,n+l\le N} e^{2\pi i(2nl\theta)}\right| \le \min\left(N, \frac{1}{\|2l\theta\|}\right),$$

we have

$$\begin{aligned} |S(\theta,\varphi)|^2 &\leq \sum_{|l|<2N} \min\left(N, \frac{1}{\|l\theta\|}\right) \\ &\leq \underset{(l=0)}{N} + 2\sum_{1\leq l\leq 2N} \min\left(N, \frac{1}{\|l\theta\|}\right). \end{aligned}$$

We shall make use of a bound for

$$\sum_{M\leq n\leq M+q} \min\left(N, \frac{1}{\|n\theta\|}\right).$$

By writing $\theta = \frac{a}{q} + \gamma$, $|\gamma| \leq \frac{1}{q^2}$, we are led to the following

Lemma 4.8.

$$\sum_{M\leq n\leq M+q} \min\left(N, \frac{1}{\|n\theta\|}\right) \leq N + q\log q$$

Proof of lemma.

Claim. *For any real number u, there exists at most three choices of n in $M \leq n \leq M+q$ such that*

$$\|n\theta - u\| \leq \frac{1}{2q}.$$

Proof of claim. Let $n = M + m$, and observe that

$$\begin{aligned} \frac{1}{2q} &\geq \|n\theta - u\| \\ &= \|(M+m)\theta - u\| \\ &= \|m\theta - (M\theta - u)\| \\ &= \|m\theta - v\|, \end{aligned}$$

whence we conclude that

$$\|n\theta - u\| \leq \frac{1}{2q}$$

implies

$$\|m\theta - v\| \leq \frac{1}{2q}.$$

We now apply the above calculation to obtain

$$\left\|m\frac{a}{q} - v\right\| = \|m\theta - m\gamma - v\| \leq \|m\theta - v\| + \|m\gamma\|.$$

Now, we have $\|m\theta - v\| \leq \frac{1}{2q}$ and $\|m\gamma\| \leq q \cdot \frac{1}{q^2} = \frac{1}{q}$, and so

$$\left\|m\frac{a}{q} - v\right\| \leq \frac{3}{2q}.$$

Therefore, there are only three choices of m, whence there are only three choices of n. □

We now let

$$\mathscr{S}=\left\{\frac{1}{q},\frac{2}{q},\frac{3}{q},\dots\frac{q/2}{q}\right\},$$

whose cardinality is $q/2$. We average the sum

$$\sum_{M\le n\le M+q}\min\left(N,\frac{1}{\|n\theta\|}\right)$$

over $\mathscr{S}$ as follows:

$$\sum_{M\le n\le M+q}\min\left(N,\frac{1}{\|n\theta\|}\right)=\frac{1}{|\mathscr{S}|}\sum_{u\in\mathscr{S}}\sum_{M\le n\le M+q}\min\left(N,\frac{1}{\|n\theta\|}\right).$$

It then follows that

$$\begin{aligned}\sum_{M\le n\le M+q}\min\left(N,\frac{1}{\|n\theta\|}\right) &= \frac{1}{|\mathscr{S}|}\sum_{u\in\mathscr{S}}\sum_{M\le n\le M+q}\min\left(N,\frac{1}{\|n\theta\|}\right)\\ &\le \frac{1}{\mathscr{S}}\sum_{u\in\mathscr{S}}\left(3N+\sum_{\substack{M\le n\le M+q\\ \|n\theta-u\|\le\frac{1}{2q}}}\frac{1}{\|n\theta\|^{-1}}\right)\\ &\le 3N+\frac{1}{|\mathscr{S}|}\sum_{u\in\mathscr{S}}\sum_{\substack{n\\ \|n\theta-u\|\le\frac{1}{2q}}}\frac{1}{\|u\|-1/2q},\end{aligned}$$

where the last inequality follows from that $u=b/q$ and $1\le b\le q/2$.

We conclude that

$$\begin{aligned}\sum_{M\le n\le M+q}\min\left(N,\frac{1}{\|n\theta\|}\right) &\le 3N+\frac{q}{q/2}\sum_{m=1}^{q/2}\frac{1}{m/q-1/2q}\\ &\le 3N+q\sum_{m=1}^{q/2}\frac{1}{m}\\ &\approx N+q\log q,\end{aligned}$$

as was to be shown. ☐

It thus follows that

$$|S(\theta,\varphi)|^2\le N+\left(\frac{N}{q}+1\right)(N+q\log q),$$

which is the desired bound. ☐

We merely state the general Weyl bound:

Theorem 4.9 (Weyl bound, degree k)**.** *Let $\theta\in\mathbb{R}$ and suppose that $(a,q)=1$ such that*

$$\left|\theta-\frac{a}{q}\right|\le\frac{1}{q^2}.$$

Then, for

$$S=\sum_{n=1}^{N}e^{2\pi iP(n)\theta}$$

with a degree-k polynomial P, we have the following bound

$$|S| \le c_{\varepsilon,k} N^{1+\varepsilon} \left(\frac{1}{N} + \frac{1}{q} + \frac{q}{N^k} \right)^{\frac{1}{2^{k-1}}} .$$

4. Major and minor arcs

We now return to the problem of Fourier multiplier:

$$m(\theta,\varphi) = \sum_{m \neq 0} \frac{2\pi i(n^2\theta + n\varphi)}{n}.$$

We set $K(x) = \frac{1}{x}$ for $|x| \ge 1$, and write

$$k = \sum_{j=0}^{\infty} K_j(x),$$

where

$$K_j(x) = \frac{\chi_{2^j \le |x| \le 2^{j+1}}(x)}{x} /$$

Then

$$m(\theta,\varphi) = \sum_{j=0}^{\infty} \left[\sum_{2^j \le |n| 2^{j+1}} \frac{e^{2\pi i(n^2\theta + n\varphi)}}{n} \right].$$

We fix $N = 10 \cdot 2^j$, j and $0 < \varepsilon \le 1/4$. Given any Θ and $N \ge 1$, Theorem 4.5 implies that there are $1 \le a \le q \le N$ such that

$$\left| \theta - \frac{a}{q} \right| \le \frac{1}{qN}.$$

If $q \le N^{1-\varepsilon}$, then we say that θ is in a *major arc*: for each $1 \le a \le q \le N^{1-\varepsilon}$, the set

$$\mathcal{M}_j \left(\frac{a}{q}, \frac{b}{q} \right) = \left\{ (\theta, q) \in [0,1]^2 : \left| \theta - \frac{a}{q} \right| \le \frac{1}{qN} \text{ and } \left| \varphi - \frac{b}{q} \right| \le \frac{1}{2q} \right\}$$

is called the *major arc*. For fixed a and q, a *minor arc* is the complement of the union of major arcs.

Proposition 4.10. *For a fixed j, all major arcs are pairwise-disjoint.*

Proof. Suppose

$$\mathcal{M}_j \left(\frac{a}{q}, \frac{b}{q} \right) \cap \mathcal{M}_j \left(\frac{a'}{q'}, \frac{b'}{q'} \right) \neq \varnothing.$$

Then there exists at least one θ such that

$$\frac{1}{qq'} \le \left| \frac{aq' - q'q}{qq'} \right| = \left| \frac{a}{q} - \frac{a'}{q'} \right| \le \left| \frac{a}{q} - \theta \right| + \left| \frac{a'}{q'} - \theta \right| \le \frac{1}{qN} + \frac{1}{q'N}.$$

Suppose $q \le q'$. Then we have

$$\frac{1}{qq'} \le \frac{2}{qN},$$

and so

$$q' \ge \frac{N}{2};$$

this is evidently absurd, for $q' \le N^{1-\varepsilon}$. $\square$

We now define a new version, independent of j:

$$I\left(\frac{a}{q},\frac{b}{q}\right)=\left\{(\theta,\varphi):\left|\theta-\frac{a}{q}\right|\le\frac{1}{10q^2},\ \left|\varphi-\frac{b}{q}\right|\le\frac{1}{2q}\right\}.$$

These are disjoint in a weaker sense: If $I(\frac{a'}{q'},\frac{b'}{q'})\cap I(\frac{a}{q},\frac{b}{q})$, then q and q' must be *widely separated*. What do we mean by this? Recall that

$$\frac{1}{qq'}\le\left|\frac{aq'-a'q}{qq'}\right|=\left|\frac{a}{q}-\frac{a'}{q'}\right|\le\left|\theta-\frac{a}{q}\right|+\left|\theta-\frac{a'}{q'}\right|\le\frac{1}{10q^2}+\frac{1}{p(q')^2}$$

if and only if

$$10\le\left(\frac{q'}{q}+\frac{q}{q'}\right),$$

which then implies that either $q\ge 5q'$ or $q'\ge 5q$, hence "widely separated." We remark that

$$\mathcal{M}_j\left(\frac{a}{q},\frac{b}{q}\right)\subseteq I\left(\frac{a}{q},\frac{b}{q}\right).$$

We now assume that θ,φ lie in a minor arc. We use partial summation to obtain

$$\left|\sum_{|n|\approx 2^j}\frac{e^{2\pi i(n^2\theta+n\varphi)}}{n}\right|=\left|\sum_{|n|\approx 2^j}\frac{S_n-S_{n-1}}{n}\right|,$$

where

$$|S_n|=\left|\sum_{k=1}^{n}e^{2\pi i(k^2\theta+k\varphi)}\right|.$$

If $\left|\theta-\frac{a}{q}\right|\le\frac{1}{q^2}$ and $(a,q)=1$, then by Theorem 4.7 we have

$$|S_n|\le\left(nq^{-1/2}+q^{1/2}\right)(\log q)^{1/2}.$$

Since θ is in a minor arc, any such a/q must have a large q, i.e.,

$$N^{1-\varepsilon}\le q\le N.$$

We now note that

$$\left|\sum_{|n|\approx 2^j}\frac{S_n-S_{n-1}}{n}\right|=\sum_{|n|\approx 2^j}\left|\frac{1}{n}-\frac{1}{n+1}\right||S_n|=2^j\cdot 2^{-2j}\sup_{|n|\approx 2^j}|S_n|,$$

for

$$\frac{1}{n}-\frac{1}{n+1}=O\left(\frac{1}{n^2}\right).$$

In particular,

$$2^j\cdot 2^{-2j}\sup_{|n|\approx 2^j}|S_n|=2^j\cdot 2^{-2j}\left(Nq^{-12}+q^{1/2}\right)(\log q)^{1/2}.$$

Since $N=2^j\cdot 10$ and $N^{1-\varepsilon}\le q\le N$, we have

$$2^j\cdot 2^{-2j}\left(Nq^{-12}+q^{1/2}\right)(\log q)^{1/2}\le 2^{-j/2+\varepsilon' j},$$

whence the total contribution of minor arcs is

$$\sum_{j=0}^{\infty}2^{-j(\frac{1}{2}-\varepsilon')}<C,$$

which is independent of θ as long as θ is in minor arcs.

We now turn to major arcs. Let $\theta \in \mathcal{M}_j(\frac{a}{q}, \frac{b}{q})$, and write $\theta = \frac{a}{q} + \alpha$ and $\varphi = \frac{b}{q} + \beta$. We shall separate the sum

$$\sum_{|n|\approx 2^j} \frac{e^{-2\pi i(n^2(\frac{a}{q}+\alpha)+n(\frac{b}{q}+\beta))}}{n}$$

into the product of the arithmetic part and an integral. To this end, we write $n = mq + l$, where $m, l \in \mathbb{Z}$ such that $1 \leq l \leq q$. The above sum then turns into the following:

$$\sum_{1\leq l\leq q} \sum_{\frac{2^j}{q}\leq m\leq \frac{2^{j+1}}{q}} \frac{e^{-2\pi i((mq+l)^2(\frac{a}{q}+\alpha)+(mq+l)(\frac{b}{q}+\beta))}}{mq+l} + O(q2^j).$$

The error term $O(q2^j)$ is acceptable, viz.,

$$\sum_{j=0}^{\infty} q2^{-j} \leq \sum_{j=0}^{\infty} (10\cdot 2^j)^{1-\varepsilon}2^{-j} \approx \sum_{j=0}^{\infty} 2^{-j\varepsilon} < C < \infty;$$

we therefore omit the error term of type $O(q2^j)$ from now on.

Observe that

$$\begin{aligned} \left(\frac{a}{q}+\alpha\right)(mq+l)^2 &= (m^2q^2+2mql+l^2)\left(\frac{a}{q}+\alpha\right) \\ &= e^{2\pi i l^2 a/q} e^{2\pi i (mq+l)^2\alpha} \end{aligned}$$

The dyadic sum thus equals

$$\sum_{l=1}^{q} e^{[2\pi i(l^2a+lb)]/q} \sum_{|m|\approx \frac{2^j}{q}} \frac{e^{2\pi i((mq+l)^2\alpha+(mq+l)\beta)}}{mq+l}.$$

We can approximate the first sum by appealing to Theorem 4.6:

$$\left|\sum_{l=1}^{q} e^{[2\pi i(l^2a+lb)]/q}\right| = \left|S\left(\frac{a}{q}+\frac{b}{q}\right)\right| \leq cq^{1/2}.$$

We shall approximate the second sum via

Theorem 4.11 (Van der Corput)**.** *Suppose that f is a real-valued C^2-function such that*

(1) f' is monotonic
(2) $|f'(x)| \leq \gamma < 1$.

Suppose furthermore that φ is a differentiable map such that

(1) $|\varphi(x)| \leq 1$ for all x
(2) $\int_{-\infty}^{\infty} |\varphi'(x)|\,dx \leq 1$.

Then

$$\sum_{n=a}^{b} e^{2\pi i f(n)}\varphi(n) = \int_a^b e^{2\pi i f(x)}\varphi(x)\,dx + O\left(\int_a^b |\varphi'(x)|\,dx\right),$$

where the error term equals $O(1)$.

Here we set $f(x) = (xq+l)^2\alpha + (xq+l)\beta$, so that

$$\begin{aligned}
|f'(x)| &= |2\alpha q(xq+l)| + |q\beta| \\
&\leq 2q\left(\frac{1}{q^N}\right)\left(\frac{2^{j+1}}{q}\cdot q + q\right) + q\cdot\frac{1}{2q} \\
&\leq 2\left(\frac{1}{10\cdot 2^j}\right)\left(2^{j+1} + (10\cdot 2^j)^{1-\varepsilon}\right) + \frac{1}{2} \\
&< 1.
\end{aligned}$$

Moreover, we set $\varphi(x) = \frac{1}{xq+l}$, which is clearly bounded by 1, and

$$\int_{-\infty}^{\infty} |\varphi'(x)| \leq q\int_{-\infty}^{\infty} \frac{1}{(xq+l)^2} \leq \int_{-\infty}^{\infty} \frac{du}{(u+l)^2} \leq 1.$$

We can therefore apply Theorem 4.11 to conclude that our dyadic sum equals

$$\sum_{l=1}^{a} e^{[2\pi i(l^2a+lb)]/q} \int_{|y|\approx\frac{2^j}{q}} e^{2\pi i((yq+l)^2\alpha+(yq+l)\beta)}\,\frac{dy}{yq+l} + \text{ error },$$

where the error term is $O(q2^{-j})$, and so negligible.

Our approximation *still* depends on l so we are yet to be done. Note that

$$\begin{aligned}
\int_{|y|\approx\frac{2^j}{q}} e^{2\pi i((yq+l)^2\alpha+(yq+l)\beta)}\,\frac{dy}{yq+l} &= \frac{1}{q}\int_{|\frac{x-l}{q}|\approx\frac{2^j}{q}} e^{2\pi i(x^2\alpha+x\beta)}\,\frac{dx}{x} \\
&= \frac{1}{q}\int_{|x|\approx 2^j} e^{2\pi i(x^2\alpha+x\beta)}\,\frac{dx}{x} + \left(\frac{1}{q}\cdot q\cdot 2^{-j}\right).
\end{aligned}$$

The last term is $O(2^{-j})$, so we ignore it. We have thus shown that

$$\sum_{|n|\approx 2^j} \frac{e^{2\pi i(n^2\theta+n\varphi)}}{n} = \frac{1}{q}S\left(\frac{a}{q},\frac{b}{q}\right)\int_{|x|\approx 2^j} e^{2\pi i(x^2\alpha+x\beta)}\,\frac{dx}{x} + \text{ acceptable error.}$$

Recall that[1]

$$\Phi(u,v) = \int_{1\leq|x|\leq 2} e^{2\pi i(ux^2+vx)}\,\frac{dx}{dx},$$

which then yields

$$\sum_{|n|\approx 2^j} \frac{e^{2\pi i(n^2\theta+n\varphi)}}{n} = \frac{1}{q}S\left(\frac{a}{q},\frac{b}{q}\right)\Phi(2^{2j}\alpha, 2^j\beta) + \text{ acceptable error.}$$

How do we proceed from here? We shall make use of the following[2]

Lemma 4.12. *(i)* $|\Phi(u,v)| \leq A(|u|^{1/2}+|v|)$ *if* $|u|^{1/2}+|v| \leq 1$*;*
(ii) $|\Phi(u,v)| \leq A(|u|^{1/2}+|v|)^{-1/2}$ *if* $|u|^{1/2}+|v| \geq 1$.

Fix θ, φ. We shall "sum over j." Toward this end, we let I_{a_j/q_j} denote the major arc θ lies in for j (if any). We sum over all j such that $\frac{a_j}{q_j}$ is the same

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[1] Stein's notes and Lillian's notes have u and v backwards.

[2] Look at Stein's notes

fraction as $\frac{a}{q}$:

$$\sum_{\substack{0\leq j \\ \text{All } j \text{ s.t.} \\ \frac{a_q}{b_q} = \text{fixed } \frac{a}{q}}} \Phi(2^{2j}\alpha, 2^j\beta) \leq \sum_{j=0}^{\infty} |\Phi(2^{2j}\alpha, 2^j\beta)|.$$

By the above lemma, this is bounded by a constant C. We have thus shown that

$$\sum_{j=0}^{\infty} \sum_{|n|\approx 2^j} \frac{\text{oscillatory bit}(\theta)}{n} \leq \sum_{\frac{a}{q}, \frac{b}{q}} \frac{1}{q} S\left(\frac{a}{q}, \frac{b}{q}\right) \sum_{\substack{j \\ \frac{a_j}{q_j} = q}} |\Phi|,$$

where the second sum is bounded by C.

We now approximate the first sum. We can merely sum over all possible q, for the major arcs are disjoint. We observe that

$$\begin{aligned} \sum_{\frac{a}{q}, \frac{b}{q}} \frac{1}{q} S\left(\frac{a}{q}, \frac{b}{q}\right) &\leq c \sum_{q} \frac{1}{q^{1/2}} \chi_{I(\frac{a}{q}, \frac{b}{q})}(\theta) \\ &\leq c \sum_{j=0}^{\infty} (5^j)^{-1/2} \\ &\leq A, \end{aligned}$$

for $q \geq 5q'$.

CHAPTER 5

Discrete Fractional Operator (L. B. Pierce)

Definition 5.1. Given an integer $k \geq 1$ and a real number $\lambda \in (0,1)$, we define the *discrete fractional operator* to be the sum

$$I_{k,\lambda} f(n) = \sum_{m=1}^{\infty} \frac{(n - m^k)}{m^\lambda}$$

for each f.

We have shown that $I_{1,\lambda}$ is a bounded mapping from l^p to l^q if and only if

$$\frac{1}{q} \leq \frac{1}{p} - (1 - \lambda).$$

We have the following extension of this theorem:

Conjecture 5.2. *$I_{k,\lambda} : l^p \to l^q$ if and only if*

(i) $\frac{1}{q} \leq \frac{1}{p} - \frac{(1-\lambda)}{k}$

(ii) $\frac{1}{q} < \lambda$, $\frac{1}{p} > 1 - \lambda$.

$k = 1$ is easy, as we have shown above. $k = 2$ is hard, and was established in a series of papers by Stein, Wainger, Oberlin, and Ionescu. $k \geq 3$ is extremely difficult, and is, in some sense, equivalent to a 100-year-old unsolved problem in number theory.

1. Basic properties of fractional integral operator

We now investigate a few basic properties of the fractional integral operator.

Proposition 5.3. *$I_{k,\lambda}$ is translation-invariant.*

Indeed,

$$\widehat{(I_{k,\lambda} f)}(\theta) = m_{k,\lambda}(\theta) \hat{f}(\theta),$$

where

$$m_{k,\lambda}(\theta) = \sum_{m=1}^{\infty} \frac{e^{-2\pi i m^k \theta}}{m^\lambda}.$$

Theorem 5.4. *For each $0 < \lambda < 1$ and for every $\alpha > 0$,*

$$|\{\theta : m_{k,\lambda}(\theta) > \alpha\}| \leq \alpha^{-r},$$

where

$$r = \frac{k}{1 - \lambda}.$$

This means that $m_{k,\lambda} \in L^{r,\infty}[0,1]$, called either the *Lorentz space* or the *weak L^r space.*

Lemma 5.5 (Stein-Wainger)**.** *If T be a convolution operator acting on functions $f : \mathbb{Z} \to \mathbb{C}$ with Fourier multiplier m, viz.,*

$$(Tf)^n(\theta) = m(\theta)\hat{f}(\theta),$$

then $m \in L^{r,\infty}[0,1]$ implies

$$T : l^p \to l^q$$

for all

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{r}$$

and $1 < p \leq 2 \leq q < \infty$.

The proof of this lemma, which we omit, makes use of properties of Lorentz spaces. (Look it up in Pierce's thesis)

To show Theorem 5.4, it suffices to prove that $m_{k,\lambda} \in L^{r,\infty}[0,1]$ for $\lambda_k^* < \lambda < 1$, where

$$\lambda_k^* = 1 - \frac{k}{2k-1};$$

note that $\lambda_k^* \to 1/2$ as $k \to \infty$. This line of reasoning is a "very standard interpolatino argument."

There are two ways of establishing that $m_{k,\lambda} \in L^{r,\infty}[0,1]$. The first is a circle-method argument. Noting that

$$m_{k,\lambda}(\theta) = \sum_{j=0}^{\infty} \sum_{|m| \approx 2^j} \frac{e^{-2\pi i m^k \theta}}{m^\lambda},$$

we can use the Weyl bound for θ lying in a minor arc with respect to j, and an arithmetic integral for θ lying in a major arc with respect to j. The bound

$$\left| S_k\left(\frac{a}{q}\right) \right| = \left| \sum_{n=1}^{q} e^{[-2\pi i a n^k]/q} \right| \leq q^{1-1/k}$$

is the standard bound on Gauss sums. For the restricted range

$$1 - \frac{1}{2^k} < \lambda < 1,$$

we may consider

$$1 - \frac{1}{2^{k-1}+1} < \lambda < 1$$

when k is small, and

$$1 - \frac{1}{\frac{3}{2}k^2 \log k} < \lambda < 1$$

when k is large.

The second method makes use of Waring's problem and the mean-values of Weyl sums. We lower our expectations to

$$\frac{1}{2} < \lambda < 1,$$

which leads us to the following

Claim. *For $r = \frac{k}{1-\lambda}$, we have*

$$L^\infty[0,1] \subseteq \cdots \subseteq L^r[0,1] \subseteq L^{r,\infty}[0,1] \subseteq L^p[0,1]$$

whenever $r > p$.

Proof of claim. To see this, we suppose that $m \in L^{r,\infty}[0,1]$, and set

$$\Lambda(\alpha) = |\{x \in [0,1] : |m(x)| > \alpha\}| .$$

By the assumption, $\Lambda(\alpha) \leq \alpha^{-n}$, and so

$$\begin{aligned} \|m\|_{L^p[0,1]} &= \frac{1}{p}\int_0^\infty \alpha^{p-1}\Lambda(\alpha)\,d\alpha \\ &= \frac{1}{p}\int_0^1 \alpha^{p-1}\Lambda(\alpha)\,d\alpha + \frac{1}{p}\int_1^\infty \alpha^{p-1}\Lambda(\alpha)\,d\alpha. \end{aligned}$$

Note that

$$\int_0^1 \alpha^{p-1}\Lambda(\alpha)\,d\alpha \leq \int_0^1 \alpha^{p-1}\,d\alpha = O(1),$$

for $\Lambda(\alpha)$ is bounded, and

$$\int_1^\infty \alpha^{p-1}\Lambda(\alpha)\,d\alpha \leq \int_1^\infty \alpha^{p-1-r}\,d\alpha = O(1),$$

for $r > 0$. It follows that $m \in L^p[0,1]$, and we are done. ☐

What is $r = \frac{k}{1-\lambda}$ when $\frac{1}{2} < \lambda < 1$? Note that this implies $2k < r < \infty$. We wish to show that

$$m_{k,\lambda} \in L^{2k}[0,1]$$

whenever $\frac{1}{2} < \lambda < 1$.

Why might this be reasonable? We assume that $m_{k,\lambda} \in L^{2k}[0,1]$ for all $\frac{1}{2} < \lambda < 1$, which then implies that $(m_{k,\lambda}) \in L^2$. Then

$$\left(\sum_{n=1}^{\infty} \frac{e^{-2\pi i n^k \theta}}{n^\lambda}\right)^k = \sum a_l e^{-2\pi i l\theta}$$

is in $L^2[0,1]$ for all $\frac{1}{2} < \lambda < 1$, where the Fourier coefficient a_l is computed by

$$a_l = \sum_{\substack{n_1,\ldots,n_l \\ l = n_1^k + \cdots + n_k^k}} \frac{1}{n_1^\lambda \cdots n_k^\lambda}.$$

Note that $n_j^k \leq l$ implies that $n_1^\lambda \cdots n_k^\lambda \leq l^\lambda$, whence

$$a_l \geq l^{-\lambda} r_{k,k}(l).$$

Here we have written $r_{k,k}(l)$ to denote the number of representations

$$l = n_1^k + \cdots + n_k^k.$$

Since $\sum a_l e^{2\pi i l\theta} \in L^2[0,1]$, we can use Parseval's theorem to show that

$$\sum_l |a_l|^2 < \infty,$$

whence

$$\sum_{l=1}^{\infty} (r_{k,k}(l))^2 l^{-2\lambda} < \infty.$$

Applying partial summation formula to the above expression, we have

$$\sum_{l=1}^{N} (r_{k,k}(l))^2 l^{-2\lambda} = \sum_{l=1}^{N} (r_{k,k}(l))^2 N^{-2\lambda} - \int_1^N \sum_{k=1}^{N} (r_{k,k}(l))^2 (-2\lambda) u^{-2\lambda-1}\,du.$$

By the assumption, this is bounded for all $\frac{1}{2} < \lambda < 1$. In particular, we can now conclude that

$$\sum_{l=1}^{N} (r_{k,k}(l))^2 = O(N^{1+\varepsilon})$$

as $N \to \infty$.

It then follows that we have, on average,

$$r_{k,k}(l) \leq l^{\varepsilon}.$$

If we write $r_{s,k}(l)$ to denote the number of representations

$$l = x_1^k + \cdots + x_s^k,$$

we know from Waring's problem that

$$r_{s,k}(l) \sim c_{s,k}(l) l^{s/k-1} + o(l^{s/k-1})$$

for sufficiently large s with respect to k.

It is a classic result that $r_{2,2}(l) \ll l^{\varepsilon}$ is true. Mahler, however, showed in 1930 that $r_{3,3}(l) \ll l^{\varepsilon}$ is utterly false. If we look at the averages, then we have

$$\begin{aligned} \sum_{k=1}^{N} r_{k,k}(l) &= \sum_{l=1}^{N} \#\{l = x_1^k + \cdots + x_k^k\} \\ &\leq (N^{1/k})^k \\ &= N, \end{aligned}$$

which yields no information. This leads us to the following

Conjecture 5.6 (Hardy-Littlewood, Hypothesis K^*)**.** *For each* $k \geq 2$*,*

$$\sum_{l=1}^{N} (r_{k,k}(l))^2 = O(N^{1+\varepsilon})$$

as $N \to \infty$*.*

For $k = 2$, we have shown the stronger result

$$\sum_{l=1}^{N} = O(N^{1+2\varepsilon}).$$

If $k = 3$, the GRH, HW_6 of Hooley and Heath-Brown shows that the above works. We have no information for $k \geq 4$.

In the other direction, Stein and Wainger showed that Hypothesis K^* is equivalent to $m_{k,\lambda}(\theta) \in [0, 1]$ for all $\frac{1}{2} < \lambda < 1$. If, instead, we assume *property* $K^*_{k,k}(\beta)$, viz.,

$$\sum_{l=1}^{N} (r_{k,k}(l))^2 = O(N^{\beta+\varepsilon})$$

for $1 \leq \beta$ as $N \to \infty$, then something weaker holds. Indeed, $\beta \leq 2$ in this case. If we assume *property* $K^*_{s,k}(\beta)$, viz.,

$$\sum_{l=1}^{N} (r_{k,s}(l))^2 = O(N^{\beta+\varepsilon})$$

as $N \to \infty$ for $\beta \geq 1$, then, given $\beta > 0$, we have equivalence of the following statements:

- $m_{k,\lambda} \in L^{2s}[0,1]$ for all $\frac{\beta k}{2s} < \lambda < 1$
- Property $K^*_{s,k}(\beta)$ is true.

It is known that

$$r_{s,k}(l) \sim c_{s,k}(l) l^{s/k-1} + \text{ error },$$

where s is "much bigger" tha k. Wooley (2001) showed that

$$s \gg 2k^2 - 2\left[\frac{\log k}{\log 2}\right].$$

For $s = k$, Hua's inequality yields substantial information for $\beta > 1$. If s is small with respect to k ($s < k$), then

$$\sum_{l=1}^{N} (r_{s,k}(l))^2 = O(N^{s/k})$$

whenever

$$s \leq \frac{1}{4}\frac{\log k}{\log\log k};$$

this is a result of Salberger and Wooley, published in 2010.

Unfortunately, if $\deg(P) > 2$, then studying

$$\sum f(n - P(m))K(m)$$

is very, very hard.

2. Quadratic fractional integral operator

We define a "new" operator

$$I_\lambda f(n) = \sum_{\substack{m \neq 0 \\ m \in \mathbb{Z}}} \frac{f(n-m^2)}{m^\lambda},$$

which satisfies the following:

Theorem 5.7 (Stein-Wainger-Oberlin-Ionescu)**.** *For $0 < \lambda < 1$, we have $I_\lambda : l^p \to l^q$ if and only if*

(i) $\frac{1}{q} \leq \frac{1}{p} - \frac{(1-\lambda)}{2}$
(ii) $\frac{1}{q} < \lambda$ *and* $\frac{1}{p} > 1 - \lambda$*.*

The proof of this general statement is difficult and very deep. We shall instead show that $m_\lambda(\theta) \in L^{r,\infty}$, where $r = \frac{2}{1-\lambda}$. Recall that

$$m_\lambda(\theta) = \sum_{m \neq 0} \frac{e^{-2\pi i m^2 \theta}}{m^\lambda}$$

We make use of the following identity:

$$m^{-\lambda} = (2\pi)^{\lambda/2}\Gamma\left(\frac{\lambda}{2}\right)\int_0^\infty e^{-2\pi m^2 y} y^{\lambda/2-1}\, dy.$$

Appealing to the identity, we have

$$m_\lambda(\theta) = \sum_{m\neq 0} \frac{e^{-2\pi i m^2\theta}}{m^\lambda} = c_{k,\lambda} \int_0^\infty \sum_{m\neq 0} e^{-2\pi m^2(y+i\theta)} y^{\lambda/2-1}\, dy. \tag{1}$$

Here

$$\sum_{m\neq 0} e^{-2\pi m^2(y+i\theta)}$$

is a theta function.

Historically, the classical theta function were defined by

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau}$$

where $\tau = x + iy$ and $y > 0$: hence in $\mathbb{H}$. Note that Θ is a *modular form* with respect to $SL_2(\mathbb{Z})$. First off, $\Theta(\tau+2) = \Theta(\tau)$. Secondly, we shall use Poisson summation to derive an expression for $\Theta(-1/\tau)$. We define

$$f(u) = e^{i\pi u^2(iy)} = e^{-\pi u^2 y} = e^{-\pi(u\sqrt{y})^2},$$

whence

$$\begin{aligned}\hat{f}(u) &= \int_{-\infty}^{\infty} e^{-\pi(u\sqrt{y})^2} e^{-2\pi i u v}\, du \\ &= \frac{1}{\sqrt{y}} \int_{-\infty}^{\infty} e^{-\pi w^2} e^{-2\pi i w(v/\sqrt{y})}\, dw.\end{aligned}$$

Since the Fourier transform of the Gaussian $e^{-\pi x^2 y}$ is itself, we have

$$\begin{aligned}\hat{f}(u) &= \frac{1}{\sqrt{y}} \int_{-\infty}^{\infty} e^{-\pi w^2} e^{-2\pi i w(v/\sqrt{y})}\, dw \\ &= \frac{1}{\sqrt{y}} e^{-\pi(\frac{v}{\sqrt{y}})^2} \\ &= \frac{1}{\sqrt{y}} e^{-\pi v^2/y}.\end{aligned}$$

By Poisson summation, we have

$$\begin{aligned}\sum f(n) &= \sum \hat{f}(n) \\ \sum e^{i\pi n^2(iy)} &= \frac{1}{y} \sum_n e^{i\pi n^2(i/y)} \\ &= \left(\frac{i}{iy}\right)^{1/2} \sum_n e^{-\pi i n^2/(iy)}.\end{aligned}$$

If we set $\tau = iy$, then this result holds for a line on the complex plane, whence by analytic continuation this identity holds whenever it is holomorphic. Therefore, we can write

$$\Theta(\tau) = \left(\frac{i}{\tau}\right)^{1/2} \Theta(-1/\tau),$$

as was to be shown. This is called the *Jacobi inversion formula.*

Consider equation 1. $\int_1^\infty$ has absolutely convergent Fourier series, hence the corresponding operator maps l^p to l^q for all $p \geq 1$. Indeed, the Fourier coefficients of $\int_1^\infty$ are as follows:

$$\sum_m e^{-2\pi i m^2\theta} \int_1^\infty e^{-2\pi m^2 y} y^{\lambda/2-1}\, dy = \sum c_m e^{-2\pi i m^2\theta};$$

here we have

$$c_1 = \int_1^\infty e^{-2\pi y} y^{\lambda/2-1}\, dy = O(1)$$

for $|m| = 1$, and

$$\begin{aligned} |c_m| &\leq \int_1^\infty e^{-2\pi m^2 y} e^{2\pi y}\, dy \\ &\leq c \int_1^\infty e^{-2\pi(|m|^2-1)y}\, dy \\ &= O\left(\frac{e^{-|m|^2-1}}{|m|^2-1}\right) \end{aligned}$$

for $|m| \geq 2$, where the error term is rapidly decreasing.

We now establish the second part of our assertion. Given a multiplier m with absolutely convergent series, we suppose that T has a multiplier m. Then $Tf = f * K$, where $K = m^\vee$, and so

$$K(n) = \int_0^1 m(\theta) e^{2\pi i n\theta}\, d\theta.$$

By Young's inequality, we have $\|Tf\|_{l^p} \leq \|k\|_{l^1} \|f\|_{l^p}$, whence

$$\begin{aligned} \|K\|_{l^1} &= \sum |K(n)| \\ &= \sum_n \left| \int_0^1 m(\theta) e^{2\pi i n\theta}\, d\theta \right| \\ &= \sum_n |c_{-n}(m)| \\ &\leq O(1). \end{aligned}$$

It follows that $T : l^p \to l^q$ for all $p \leq q$.

We reduce equation 1 further by noting that it is sufficient to study the Fourier multiplier of

$$\nu_\lambda(\theta) = \int_0^1 \Theta(y + i\theta) y^{\lambda/2-1}\, dy,$$

where

$$\Theta(z) = \sum_{m\in\mathbb{Z}} e^{-2\pi m^2 z}.$$

This series is absolutely convergent for $\mathrm{Re}(z) > 0$, and uniformly convergent for $\mathrm{Re}(z) \geq \delta > 0$ for every $\delta > 0$. Note that we have added the $m = 0$ term to

$$\sum_{m\neq 0} e^{-2\pi m^2(y+i\theta)}.$$

We now consider theta functions associated to a positive-definite quadratic form Q

$$\Theta_Q(z) = \sum_{m \in \mathbb{Z}^k} e^{-2\pi Q(m) z},$$

where $\mathrm{Re}(z) > 0$. Here

$$Q(x_1, \dots, x_k) = x_1^2 + \cdots + x_k^2 = |x|^2.$$

It is convenient to write

$$Q(x) = \frac{1}{2} x^t A x,$$

where

$$A = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}.$$

A is a symmetric positive-definite matrix with even diagonal entries. We can then define the *adjoint quadratic form*

$$Q^*(x) = \frac{1}{2} x^t A^{-1} x.$$

We remark that there is a transformation law between $\Theta_Q(z)$ and $\Theta_{Q^*}(1/z)$. More precisely, the following holds:

Theorem 5.8. *For $\theta = \frac{a}{q} + \alpha$, we have*

$$\Theta_Q(y + i\theta) = \frac{1}{q^k |A| (y + i\alpha)^{k/2}} \sum_{m \in \mathbb{Z}^k} S_q(a, -m; q) e^{-\frac{2\pi Q^*(m)}{q^2 (y + i\alpha)}},$$

where A is the matrix for Q and

$$S_Q(a, b; q) = \sum_{\substack{r \in (\mathbb{Z}/q\mathbb{Z})^k \\ r = (r_1, \dots, r_k)}} e^{2\pi i [a Q(r) + b \cdot r]/q}.$$

Proof. We shall make use of the following

Lemma 5.9. *If $(a, q) = 1$, then*

$$|S_Q(a, b; q)| \leq c q^{k/2}$$

This is, of course, a straightforward higher-dimensional generalization of the bound on the usual Gauss sums.

We claim that

$$\sum_{l \in \mathbb{Z}^k} e^{-2\pi Q(lq + r) z} = \frac{1}{q^k |A|^{1/2} z^{k/2}} \sum_{l \in \mathbb{Z}^k} e^{2\pi i r \cdot l / q} e^{-\frac{2\pi Q^*(l)}{q^2 z}}$$

whenever $\mathrm{Re}(z) > 0$. To this end, we assume without loss of generality that $z = y$ and $\mathrm{Re} > 0$; indeed, a simple analytic continuation will do. By Poisson summation, we see that

$$\begin{aligned} \sum e^{-2\pi Q(nq + r) y} &= \sum f(n) \\ &= \sum \hat{f}(n) \\ &= \frac{e^{2\pi i r \cdot n / q}}{q^k |A|^{1/2} y^{k/2}} \cdot e^{-2\pi Q^*(n / q\sqrt{y})}. \end{aligned}$$

The Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^k} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

is computed by first plugging in f and diagonalizing Q; then we note that

$$Q(x) = \sum_{i=1}^{k} \gamma_i x_i^2,$$

and split the integral: we will then end up with normalized Gaussians, whose Fourier transform we know.

We apply our claim to conclude that

$$\begin{aligned}\Theta_Q\left(y + i\left(\frac{a}{q} + \alpha\right)\right) &= \sum_m e^{-2\pi Q(m)(y+i(\frac{a}{q}+\alpha))} \\ &= \sum_{r\in(\mathbb{Z}/q\mathbb{Z})^k} \sum_{l\in\mathbb{Z}^k} e^{-2\pi Q(lq+r)(y+i(\frac{a}{q}+\alpha))}.\end{aligned}$$

Note indeed that

$$Q(u+v) = \frac{1}{2}(u+v)^t A(u+v) = Q(u) + Q(v) + u^t A v,$$

whence

$$\begin{aligned}Q(lq+r) &= Q(lq) + Q(r) + ql^t Ar \\ &= q^2 Q(l) + Q(r) + ql^t Ar.\end{aligned}$$

It thus follows that

$$\begin{aligned}\Theta_Q\left(y + i\left(\frac{a}{q} + \alpha\right)\right) &= \sum_m e^{-2\pi Q(m)(y+i(\frac{a}{q}+\alpha))} \\ &= \sum_{r\in(\mathbb{Z}/q\mathbb{Z})^k} \sum_{l\in\mathbb{Z}^k} e^{-2\pi Q(lq+r)(y+i(\frac{a}{q}+\alpha))} \\ &= \sum_{r\in(\mathbb{Z}/q\mathbb{Z})^k} e^{-2\pi i Q(r)a/q} \sum_{l\in\mathbb{Z}^k} e^{-2\pi Q(lq+r)(y+i\alpha)}.\end{aligned}$$

What we have shown in our simple case is as follows: if

$$\begin{aligned}A &= \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix} \\ Q(m) &= m^2 \\ A^{-1} &= \begin{pmatrix} 1/2 & & & \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/2 \end{pmatrix}. \\ Q^*(m) &= \frac{1}{4} m^2.\end{aligned}$$

For $\theta = \frac{a}{q} + \alpha$, we have

$$\Theta(y + i\theta) = \frac{1}{q2^{1/2}(y + i\alpha)^{1/2}} \cdot \sum_{m\in\mathbb{Z}} S(a, -m; q) e^{-\frac{\pi m^2}{4q^2(y+i\alpha)}},$$

where

$$S(a, b; q) = \sum_{r \pmod{q}} e^{-2\pi i r^2 a/q} e^{-2\pi i b r/q}.$$

Indeed, $S(a/q) = S(a; q) = S(a, 0; q)$.

We shall show that

$$\sum f(n) = \sum \hat{f}(n) = \hat{f}(0) + \sum_{n\neq 0} \hat{f}(n),$$

where $\sum_{n\neq 0} \hat{f}(n)$ is a small error term. To this end, we assume that

(1) $(a, q) = 1$, $1 \leq a \leq q$
(2) $q \leq y^{-1/2}$
(3) $q|\alpha| \leq y^{1/2}$, $\theta = \frac{a}{q} + \alpha$.

and make use of the following

Lemma 5.10 (Approximation identity)**.**

$$\Theta(y + i\theta) = \frac{S(a; q)}{q\sqrt{2}}(y + i\alpha)^{-1/2} + (y^{-1/4}).$$

Proof of lemma. We note that

$$\frac{S(a; q)}{q\sqrt{2}}(y + i\alpha)^{-1/2}$$

is the $m = 0$ term, whence it suffices to show that $(y^{-1/4})$ corresponds to the remaining sum $\sum_{m\neq 0}$. We observe that

$$\left| \sum_{m\neq 0} \frac{|S|}{q\sqrt{2}(y + i\alpha)^{1/2}} e^{-\frac{-\pi m^2}{4q^2(y+i\alpha)}} \right| \quad \leq \quad \frac{c}{q^{1/2}(y^2 + \alpha^2)^{1/4}} \sum_{m\neq 0} e^{-\frac{\pi m^2 y}{4q^2(y^2+\alpha^2)}}.$$

By the assumptions, we have $q^2y^2 \lesssim y$ and $\alpha^2 q^2 \lesssim y$, and so

$$1 \lesssim \frac{y}{q^2(y^2 + \alpha^2)}.$$

It thus follows that

$$\begin{aligned} \sum_{m\neq 0} &\leq c \sum_{m\neq 0} e^{-Cm^2 u} \\ &\leq Ce^{-Cu} \\ &\leq u^{-1/4}, \end{aligned}$$

where $1 \lesssim u$, whence

$$\begin{aligned} \text{total error term} &\leq \frac{C}{q^{1/2}(y^2 + \alpha^2)^{1/4}} \left(\frac{y}{q^2(y^2 + \alpha^2)} \right)^{-1/4} \\ &= O(y^{-1/4}). \end{aligned}$$

□

We now rewrite the desired integral as

$$\int_0^1 \Theta(y+i\theta)y^{\lambda/2-1}\,dy \quad = \quad \sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{-j+1}} \Theta(y+i\theta)y^{\lambda/2-1}\,dy.$$

We define the major arcs for $2^{-j} \le y \le 2^{-j+1}$ as follows: we fix j, and, for each $1 \le q \le \frac{1}{10}2^{j/2}$, $1 \le a \le q$, and $(a,q)=1$, we set

$$\mathcal{M}_j(a/q) = \left\{\theta \in [0,1] : \left|\theta - \frac{a}{q}\right| \le \frac{1}{q2^{j/2}}\right\}.$$

The corresponding minor arcs for j are

$$[0,1] \smallsetminus \bigcup_{a,q} \mathcal{M}_j(a/q).$$

Of course, the major arcs are disjoint: if $(a,q) \neq (a',q')$, then

$$\frac{1}{qq'} \le \left|\frac{a}{q} - \frac{a'}{q'}\right| \le \left|\theta - \frac{a}{q}\right| + \left|\theta - \frac{a'}{q'}\right| \le \frac{1}{q2^{j/2}} + \frac{1}{q'2^{j/2}}.$$

It is again convenient to have a j-independent version of the major arcs. To this end, we define, for $(a,q)=1$,

$$\mathcal{M}^*(a/q) = \left\{\theta : \left|\theta - \frac{a}{q}\right| \le \frac{1}{10q^2}\right\}.$$

$\mathcal{M}^*(a/q)$ has the following properties:

(1) $\mathcal{M}_j(a/q) \subseteq \mathcal{M}^*(a/q)$.
(2) $\mathcal{M}^*(a/q) \cap \mathcal{M}^*(a'/q')$ if $q' \le a \le 2q'$.

There are three types of terms in $\nu_\lambda(\theta)$:

(1) The main term of Θ on major arcs
(2) The remainder term of Θ on major arcs
(3) Θ on the minor arcs.

We classify (3) first. For a fixed j, the minor arcs are

$$\left\{\theta : \text{ if } \left|\theta - \frac{a}{q}\right| \le \frac{1}{q2^{j/2}}, \text{ then } q \ge \frac{1}{10}2^{j/2}\right\}.$$

Therefore,

$$\begin{aligned}
|\Theta(y+i\theta)| &\le c\frac{|S(a;q)|}{q}\,|(y+i\alpha)|^{-1/2} + O(y^{-1/4}) \\
&\le q^{-1/2}y^{-1/2} + O(y^{-1/4}) \\
&\le c2^{-j/4}2^{j/2} + O(2^{j/4}) \\
&= O(2^{j/4}) + O(2^{j/4}) \\
&= O(2^{j/4}).
\end{aligned}$$

Plugging into $\nu_\lambda(\theta)$, we have

$$\sum_{j=1}^{\infty}\chi_{\min_j}(\theta)\int_{2^{-j}}^{2^{-j+1}} \Theta(y+i\theta)y^{\lambda/2-1}\,dy = \sum_{j=1}^{\infty}\chi_{\min_j}(\theta)\int_{2^{-j}}^{2^{-j+1}} 2^{j/4}y^{\lambda/2-1}\,dy,$$

whence

$$\sum_{j=1}^{\infty} 2^{j/4} 2^{-j(\frac{\lambda}{2})} < \infty$$

if $\lambda > \frac{1}{2}$. Therefore, we have shown that $\nu_{\lambda_{\min}}(\theta) \in L^{\infty}[0,1]$. Furthermore,

$$\sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} y^{\lambda/2-1}\, dy = O(1)$$

if $\lambda > \frac{1}{2}$, whence $\nu_{\lambda_{\text{remainder}}}(\theta) \in L^{\infty}[0,1]$.

We now consider (1), the main term of the theta function on the major arcs. Let $\chi_{a/q}^{j}$ be the characteristic function of $\mathcal{M}_j(a/q)$, and $\chi_{a/q}$ the characteristic function of $\mathcal{M}^*(a/q)$. We fix an a/q pair and sum over all j (with a fixed θ): the contribution of the main term of Θ is

$$\frac{1}{q\sqrt{2}} S(a;q) \sum_{j=1}^{\infty} \chi_{a/q}^{j}(\theta) \int_{2^{-j}}^{2^{-j+1}} \left| y + i\left(\theta - \frac{a}{q}\right)\right|^{-1/2} y^{\lambda/2-1}\, dy.$$

Observe that

$$\begin{aligned} &\left| \frac{1}{q\sqrt{2}} S(a;q) \sum_{j=1}^{\infty} \chi_{a/q}^{j}(\theta) \int_{2^{-j}}^{2^{-j+1}} \left| y + i\left(\theta - \frac{a}{q}\right)\right|^{-1/2} y^{\lambda/2-1}\, dy \right| \\ \leq \quad & cq^{-1/2} \chi_{a/q}(\theta) \int_0^1 y^{\lambda/2-1} \left| y + i\left(\theta - \frac{a}{q}\right)\right|^{-1/2} dy. \end{aligned}$$

Setting $\alpha = \theta - a/q$, we make the change of variable

$$\int_0^1 y^{\lambda/2-1} \left| y + i\left(\theta - \frac{a}{q}\right)\right|^{-1/2} dy. = |\alpha|^{\lambda/2-1/2} \int_0^{\infty} y^{\lambda/2-1} |y+i|^{-1/2}\, dy.$$

We note that

$$\int_0^1 y^{\lambda/2-1} |y+i|^{-1/2}\, dy \leq \int_0^1 y^{\lambda/2-1}\, dy = O(1),$$

for $\lambda > 1$ and $|y+i| \geq 1$. Furthermore,

$$\int_1^{\infty} y^{\lambda/2-1} |y+i|^{-1/2}\, dy \leq \int_1^{\infty} y^{\lambda/2-1} y^{-1/2}\, dy = O(1),$$

since $\lambda < 1$ and $|y+i| \geq y$. It thus follows that the contribution to the Fourier multiplier from a fixed a/q pair is

$$q^{-1/2} \chi_{a/q}(\theta) \left| \theta - \frac{a}{q} \right|^{-\frac{1}{2}(1-\lambda)}.$$

Therefore, we can write the Fourier multiplier as

$$\sum_{s=0}^{\infty} \sum_{2^s \leq q < 2^{s+1}} \sum_{\substack{(a,q)=1 \\ 1 \leq a \leq q}} q^{-1/2} \chi_{a/q}(\theta) \left| \theta - \frac{a}{q} \right|^{-\frac{1}{2}(1-\lambda)}.$$

We check that $m \in L^{r,\infty}[0,1]$. Indeed

$$|\{\theta : |m(\theta)| > \beta\}| < \beta^{-r}.$$

Note that $g(u) = |u|^{-\frac{1}{2}(1-\lambda)} \in L^{r,\infty}[0,1]$:

$$\left|\left\{u : |u| < \beta^{-\frac{2}{1-\lambda}}\right\}\right| < \beta^{-r},$$

where $r = \frac{2}{1-\lambda}$.

We shall make use of the following lemma

Lemma 5.11. *Given N functions $f_1, \ldots, f_N$ with disjoint supports and f_j uniformly in $L^{r,\infty}[0,1]$, then*

$$F_N = N^{-1/r} \sum_{j=1}^{N} f_j$$

belongs to $L^{r,\infty}[0,1]$ uniformly in N.

Proof of lemma. Note that

$$\{|F_n| > \alpha\} = \bigcup_{j=1}^{N} \{N^{-1/r}|f_j| > \alpha\}.$$

Therefore,

$$\begin{aligned} |\{|F_N| > \alpha\}| &= \sum_{j=1}^{N} |\{N^{-1/r}|f_j| > \alpha\}| \\ &\leq \sum_{j=1}^{N} \left(N^{-1}\alpha^{-r}\right) \\ &\leq \alpha^{-r} \end{aligned}$$

uniformly in N. □

We now apply the above lemma to

$$\sum_{s=0}^{\infty} \sum_{2^s \leq q < 2^{s+1}} \sum_{\substack{(a,q)=1 \\ 1 \leq a \leq q}} q^{-1/2} \chi_{a/q}(\theta) \left|\theta - \frac{a}{q}\right|^{-\frac{1}{2}(1-\lambda)}.$$

The $L^{r,\infty}$ norm of the above is

$$\begin{aligned} O(n^{1/r} 2^{-s/2}) &= O\left((2^{2s})^{\frac{1-\lambda}{2}} 2^{-s/2}\right) \\ &= O\left(2^{s(\frac{1}{2}-\lambda)}\right). \end{aligned}$$

Our multiplier has $L^{r,\infty}[0,1]$ norm

$$\sum_{s=0}^{\infty} O\left(2^{s(\frac{1}{2}-\lambda)}\right) = O(1)$$

if $\lambda > 1/2$. This concludes the proof. □

CHAPTER 6

Applications (L. B. Pierce)

The last chapter serves as a quick survey of some applications of the theory we have developed thus far. Details will be sparse.

1. Ergodic Theory

Let (X, μ) be a measure space of total measure 1 and $T : X \to X$ an invertible map. We assume furthermore that T is *measure-preserving*, viz., $\mu(T^{-1}(E)) = \mu(E)$ for each measurable subset E of X. T is said to be *ergodic* if the measure of each measurable set $E \subseteq X$ with the property $T^{-1}(E)$ is either 0 or 1.

Example. Let $X = \mathbb{R}/\mathbb{Z}$ and μ the canonical rotation-invariant measure on the circle. Then $x \mapsto x + \theta$ is ergodic if and only if θ is irrational.

The so-called *ergodic theorems* concern the *time average*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

and the *space average*

$$\frac{1}{\mu(X)} \int_X f \, d\mu.$$

Here is one such example:

Theorem 6.1 (Pointwise ergodic theorem). *Let $T : X \to X$ be an invertible, measure-preserving map. Then*

1. *the time average exists almost everywhere and the limit F is a T-invariant $L^1(X)$-function, viz., $F \circ T = F$. Furthermore, if $0 < \mu(X) < X$, then $\int F = intf$.*
2. *If T is ergodic, then F equals the space average.*

We remark that analogues L^p-theorems exist. (2) follows from (1) via the following

Lemma 6.2. *T is ergodic if and only if all T-invariant measurable functions are constant almost everywhere.*

We now consider a discrete averaging operator defined to be

$$(A_r f)(x) = \frac{1}{2r+1} \sum_{|m| \le r} f\left(T^{P(m)} x\right)$$

for each polynomial $P : \mathbb{Z} \to \mathbb{Z}$. The pointwise ergodic theorem for A_r is given by Bourgain:

Theorem 6.3 (Bourgain, 1980's)**.** *If $f \in L^p(X)$, then there exists a function $F \in L^p(X)$ such that $A_r f \to F$ pointwise almost everywhere and in L^p as $r \to \infty$. Furthermore, if T is ergodic, then*

$$F = \frac{1}{\mu(X)} \int f \, d\mu.$$

The above theorem is a consequence of the following discrete maximal inequality:

Theorem 6.4. *The discrete operator*

$$Mf(n) = \sup_{r>0} \frac{1}{2r+1} \sum_{|m| \leq r} |f(n-m)|$$

is $l^p \to l^p$ bounded for all $1 < p \leq \infty$.

The proof of the above result is analogous to the proofs we have studied thus far—for example, the circle method gets used here. Indeed, the operator obtained by dropping the supremum and fixing an $r > 0$ has the Fourier multiplier

$$\frac{1}{2r+1} \sum_{|m| \leq r} e^{-2\pi i P(m)\theta}.$$

We also remark that similar results exist for operators

$$f(x) \mapsto \frac{1}{2r+1} \sum_{|m| \leq r} f\left(T^{P_m(m)} x\right),$$

where $P_m = T^m P$. Also relevant are the operators

$$f(x) \mapsto \frac{1}{2r+1} \sum_{|m| \leq r} f\left(T^{P(m,n)} x\right),$$

which are not translation-invariant.

2. Partial Differential Equations

Consider the non-linear Schrödinger operator

$$\Delta_x u + i\partial_t u + u|u|^{p-2} = 0$$

in a periodic setting, where $u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is periodic of period one in x and with initial condition $u(x,0) = \phi(x)$. While the basic inequality to consider for the non-periodic case is the *Strichartz inequality*

$$\|e^{it\Delta}\phi\|_{L^{2(d+2)/d}(\mathbb{R}^{d+1})} = \left\| \int \hat{\phi}(\xi) e^{2\pi i (x\cdot\xi + t|\xi|^2)} \, d\xi \right\|_{L^{2(d+2)/d}(\mathbb{R}^{d+1})} \leq c\|\phi\|_{L^2(\mathbb{R}^d)},$$

the periodic case calls for the estimate

$$\left\| \sum_{\substack{|n| \leq N \\ n \in \mathbb{Z}^d}} a_n e^{2\pi i (n\cdot x + t|n|^2)} \right\|_{L^{2(d+2)/d}(\mathbb{T}^{d+1})} \leq C\|(a_n)_{n=-N}^N\|_{l^2} = C\left(\sum_{|n| \leq N} |a_n|^2\right)^{1/2},$$

where a_n is the nth Fourier coefficient of ϕ. This calls for an analysis in the discrete setting.

3. Convergernce of Fourier Series

We now turn to the oldest application of discrete analogues of harmonic analysis. Consider the partial sum

$$(S_n f)(\theta) = \sum_{n=0}^{N} \hat{f}(n) e^{2\pi i n\theta},$$

where f is 1-periodic and

$$\hat{f}(n) = \int_0^1 f(\theta) e^{-2\pi i n\theta}\, d\theta$$

for each $0 \le n \le N$. The classical question concerns the hypotheses that guarantee the convergence $S_N \to f$. Let us consider a variant of this question by looking at periodic functions f such that $\hat{f}(n) = 0$ for all n except for the terms in a sequence $(n_k)_{k=1}^{\infty}$. Then the Fourier series of f is

$$\sum_{k=0}^{\infty} \hat{f}(n_k) e^{2\pi i n_k \theta}.$$

We can now ask the similar convergence question: which hypotheses on the sequence $(n_k)_{k=1}^{\infty}$ guarantee uniform convergence of $S_N f \to f$ for all such f? Arkhipov and Oskolkov observed that the key bound is

$$\left| \sum_{1 \le k \le M} \frac{e^{2\pi i n_{k+N}\theta}}{n} \right| \le A.$$

which is uniform in M, N, and $\theta \in [0,1]$. This bound can be obtained by studying the Fourier multiplier of the discrete Hilbert transform

$$\sum_{\substack{-\infty < k < \infty \\ k \neq 0}} \frac{f(n - S(k))}{k}.$$

4. Number Theory

Consider the operator

$$(\mathcal{A}_* f)(x) = \sup_{0<\lambda<\infty} \frac{1}{S(\lambda)} \int_{|y|=\lambda} |f(x-y)|\, dy.$$

Theorem 6.5 (Stein, Stein-Wainger, Bourgain)**.** *$\mathcal{A}_*$ is $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ bounded if and only if $k \ge 2$ and $p > \frac{k}{k-1}$.*

The proof of the above theorem spans three papers. We note that an L^p-function will remain in L^p if its values are changed on a set of measure zero: in particular, $|y| = \lambda$ is of measure zero. The proof relies heavily on the notion of Fourier transform on surface with curvature.

The discrete analogue is

$$(A_* f)(x) = \sup_{r>0} \frac{1}{N(r)} \sum_{\substack{|m|=r \\ m \in \mathbb{Z}^k}} |f(n-m)|,$$

where $N(r) = \#\{m \in \mathbb{Z}^k : |m|^2 = r^2\}$ and the supremum is taken over all r such that $N(r) \neq 0$.

Theorem 6.6 (Magyar-Stein-Wainger, 2002)**.** *A_* is $l^p(\mathbb{Z}^k) \to l^p(\mathbb{Z}^k)$ bounded if and only if*

(i) $k \geq 5$ and $p > \frac{k}{k-2}$ or

(ii) $k \leq 4$ and $p = \infty$.

Let us study (2) more carefully. Writing $r_{k.2}(r^2) = N(r)$, we see that $k \geq 5$ yields

$$r_{k,2}(n) \sim c_k(n) n^{k/2-1} + o(n^{k/2-1}),$$

where $c_k(n) \geq \delta > 0$. For $k = 4$ and lower, we recall that every positive integer can be written as the sum of four squares. Since this decomposition is not necessarily unique, the c_k in the above expression can sometimes be zero. Now,

$$r_{4,2}(n) = 8 \sum_{\substack{d|n \\ 4\nmid n}} d$$

and $r_{4,2}(2^{2t}) = 8(1+2) = 24$ for all t, and so setting $f(0) = 1$ and $f(n) = 1$ for all $n \neq 0$ yields

$$A_* f(n) \geq \sup_t \frac{1}{N(2^t)} \sum_{|m|=2^t} |f(n-m)| \geq \sup_t \frac{1}{N(2^t)} \geq \frac{1}{24}$$

for $|n| = 2^t$, where $N(2^t) = r_{4,2}((2^t)^2)$. This is a counterexample.

Why were we able to come up with such a counterexample? We observe that

$$\begin{aligned} \mathrm{Vol}(B_R) &= \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n; \\ \mathrm{SA}(B_R) &= \frac{d\,\mathrm{Vol}}{dR} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)} R^{n-1}. \end{aligned}$$

Normalizing with $R = N^{1/2}$, we have

$$\begin{aligned} \mathrm{Vol}(B_R) &= \frac{\pi^{n/2}}{\Gamma(n/2+1)} N^{n/2}; \\ \mathrm{SA}(B_R) &= \frac{n\pi^{n/2}}{\Gamma(n/2+1)} N^{(n-1)/2}. \end{aligned}$$

We now observe that the discrete analogues are

$$\begin{aligned} \#\{m \in \mathbb{Z}^n : |m| \leq N\} &= \mathrm{Vol}(B_{N^{1/2}}) + \text{ error }; \\ \#\{m \in \mathbb{Z}^n : |m| = N\} &\approx c_n(N) N^{(n-2)/2} + \text{ error }; \end{aligned}$$

which have different powers. Gauss studied this problem—Gauss's circle problem—and showed that the number of lattice points inside the circle is $\pi r^2 + O(r)$. Hardy and Landau gave the lower bound of $r^{1/2+\varepsilon}$ for the error term; the recent results bring down the error term to $O(r^{0.37})$.