Math 550 - Lie Algebras, Fall 2011

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Transcriber's Note

This is a transcription of the Fall 2011 lectures on Lie algebras given by professor Vladimir Retakh at Rutgers University. The class text was James Humphrey's *Introduction to Lie Algebras and Representation Theory.*

All errors in this document are those in my transcription and interpretation of the lectures. The current version of the document is available at

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January 13, 2011

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CHAPTER 1

Basic Theory of Lie Algebras

1. Definitions and Examples

There are two approaches to developing the theory of Lie algebras: the *infinitesimal Lie groups* approach, and the algebraic approach. In this course, we shall follow the latter.

Definition. A *Lie algebra* is a vector space V over a field k with a bilinear form $[\cdot,\cdot] : V \times V \to V$, called a *commutator* or a *Lie bracket*, satisfying

(i) **Antisymmetry**. $[x, y] = -[y, x]$ for all $x, y \in V$.
(ii) **Jacobi identity**. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in V$.

A Lie algebra is often denoted by the pair $L = (V, [\cdot,\cdot])$. Another symbol commonly reserved for a generic Lie algebra is $\mathfrak{g}$. Reflecting the symmetry apparent in the definition, we sometimes write the Jacobi identity as $[x, [y, z]]_{\circlearrowleft}$. We also remark that if $\operatorname{char} k \neq 2$, then antisymmetry is equivalent to the statement that $[x, x] = 0$ for all $X \in V$.

Let us now consider some simple examples. The first example is rather trivial. If L is any vector space, and the commutator is identically zero, then L is a Lie algebra, called an *abelian Lie algebra.* Another example is the exterior product in $\mathbb{R}^3$, which is a skew-symmetric form, i.e., $a \times b = -b \times a$ for all $a, b \in \mathbb{R}^3$, such that

$$i \times j = k, \quad j \times k = i, \quad k \times i = j.$$

In fact, if we have $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, then the exterior product is the determinant

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Exercise 1.1. Prove that $\mathbb{R}^3$ with the exterior product is a Lie algebra.

Here is another example of a Lie algebra.

Exercise 1.2. Prove that an associative algebra A with the commutator $[a, b] = ab - ba$ is a Lie algebra.

If $A = \operatorname{Mat}_n(k)$, then the corresponding Lie algebra is called the *general linear Lie algebra* and is denoted by $\mathfrak{gl}_n(k)$.

As a final remark, we note that the operation $(M, N) \mapsto MN + NM$ on unitary matrices gives rise to a *Jordan algebra.* We will not study Jordan algebra in this course.

2. Derivations

Let A be an associated algebra over k. Recall that a k-linear map $D : A \to A$ is a *derivation* if the Leibniz identity holds, viz.,

$$D(ab) + (Da)b + a(Db)$$

for all $a, b \in A$.

Exercise 1.3. If D_1 and D_2 are derivations, then $[D_1, D_2] = D_1D_2 - D_2D_1$ is a derivation.

Exercise 1.4. The collection $\operatorname{Der} A$ of all derivations of A with the commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ is a Lie algebra.

Here is a special case of the above proposition. Let $A = \mathbb{C}[z]$ and D a derivation. What is $D(1)$? Since

$$D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1),$$

we have

$$D(1) = 2D(1) = 0.$$

It thus follows that $D(\lambda) = 0$ for any scalar λ. Note that this calculation holds for any associative algebra.

How about $D(z^n)$? Here we have

$$D(z^n) = D(z \cdot z^{n-1}) = D(z)z^{n-1} + zD(z^{n-1}).$$

As a simple corollary, we have

$$D(z^n) = n \cdot D(z) \cdot z^{n-1},$$

which follows from the above calculation by simple induction.

If $D(z) = P(z)$ for some polynomial $P(z)$, then D can be written as

$$D = P(z) \cdot \frac{\partial}{\partial z}.$$

Given two derivations $D_1 = P_1 \frac{\partial}{\partial z}$ and $D_2 = P_2 \frac{\partial}{\partial z}$, we have the commutator

$$[D_1, D_2] = D_1D_2 - D_2D_1 = (P_1P_2' - P_1'P_2)\frac{\partial}{\partial z}$$

This defines a Lie algebra on $\operatorname{Der} \mathbb{C}[z]$, which we denote by W_1. W_1 is the algebra of vector fields on $\mathbb{C}$.

Exercise 1.5. Define W_n analogously.

3. Ideals, Subalgebras, and Homomorphisms

Definition. Let $(L, [\cdot, \cdot])$ be a Lie algebra. A vector subspace M of L is a *Lie subalgebra* if $[x, y] \in M$ for any $X, y \in M$. A vector subspace I of L is a *Lie ideal* if $[x, y] \in I$ for any $x \in I$ and $y \in I$. We say that L is *simple* if L has no proper Lie ideals.

If $L = \mathfrak{gl}_n(k)$, then the collection $\mathfrak{sl}_n(k)$ of matrices with zero trace is a Lie subalgebra, called the *special linear Lie algebra*. Furthermore,

$$I = \left\{ \begin{pmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix} \right\}$$

is a Lie ideal.

Definition. The *direct sum* of the Lie algebras $(L_1, [\cdot,\cdot]_1), \dots, (L_n, [\cdot,\cdot]_n)$ is defined to be the vector space

$$L = L_1 \oplus \cdots \oplus L_n$$

with the commutator

$$[x,y] = \begin{cases} 0 & \text{if } x \in L_i, y \in L_j, i \neq j, \\ [x,y]_i & \text{if } x, y \in L_i. \end{cases}$$

We note that any $L_i \hookrightarrow L$ is an ideal in this case.

Definition. Let I be a Lie ideal in a Lie algebra L. Given the projection map $\pi : L \to L/I$, we define the bracket in L/I as

$$\pi([x,y]) = [\pi(x), \pi(y)].$$

This turns L/I into a Lie algebra. called a *quotient Lie algebra*.

Definition. A linear map $\varphi : L_1 \to L_2$ of Lie algebras $(L_1, [\,,\,]_1)$ and $(L_2, [\,,\,]_2)$ is a *homomorphism* if

$$\varphi([x,y]_1) = [\varphi(x), \varphi(y)]_2$$

for all $x, y \in L_1$.

We remark that, if $L_1 \xrightarrow{\varphi} L_2$ is a homomorphism, then $\ker\varphi$ is an ideal in L_1, and $\operatorname{im}\varphi$ a subalgebra of L_2.

Definition. A homomorphism $\varphi : L_1 \to L_2$ of Lie algebras is an *isomorphism* if there exists a homomorphism $\psi : L_2 \to L_1$ such that $\varphi\psi = \mathrm{id}_2$ and $\psi\varphi = \mathrm{id}_1$.

Exercise 1.6. If $\varphi : L_1 \to L_2$ is a homomorphism, then

$$L_1/\ker\varphi \cong \operatorname{im}\varphi.$$

4. Classifying complex Lie algebras of small dimensions

We now turn to the problem of classifying Lie algebras of small dimensions over $\mathbb{C}$. Clearly, $\dim L = 0$ is trivial. If $\dim L = 1$, then $L = \mathbb{C}e$ for some nonzero vector e. Since $[e,e] = 0$, we see that L is abelian.

4.1. Two-dimensional complex Lie algebras. Let $\dim L = 2$. We define the *commutator* of the Lie algebra L to be the span of all commutators $[x,y]$. We claim that $[L,L]$ is either zero- or one-dimensional in this case. To see this, we take a basis $\{f_1, f_2\}$ in L and observe that

$$[\alpha_1 f_1 + \alpha_2 f_2, \beta_1 f_1 + \beta_2 f_2] = (\alpha_1\beta_2 - \alpha_2\beta_1)[f_1, f_2].$$

Therefore, the commutator is one-dimensional if $[f_1, f_2] \neq 0$ and zero-dimensional otherwise.

If $\dim[L,L] = 0$, then L is abelian. We may thus assume that $\dim[L,L] = 1$. We take e_1 as a basis vector in $[L,L]$ and pick $\tilde{e}_2 \in L \setminus [L,L]$. Then

$$[e_1, \tilde{e}_2] = \lambda e_1$$

for some nonzero scalar λ, as (1) the commutator belongs to the one-dimensional space $[L,L]$, and (2) L is nonabelian. We take

$$e_2 = \frac{\tilde{e}_2}{\lambda},$$

so that $[e_1, e_2] = e_1$. It follows that L is isomorphic to the Lie algebra with basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_1$.

4.2. Three-dimensional complex Lie algebras. We now let $\dim L = 3$. The most important example is $\mathfrak{sl}_2(\mathbb{C})$, which is generated by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the commutator defined by

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Exercise 1.7. Check that the above construction defines a Lie algebra.

Another important example is the *Heisenberg algebra* H, generated by

$$\begin{aligned} x &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ z &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

with the commutator defined by

$$[x, y] = z, \quad , [x, z] = 0, \quad , [y, z] = 0.$$

Exercise 1.8. Check that the above construction defines a Lie algebra.

We now return to the problem of classifying Lie algebras of dimension 3. If $\dim[L, L] = 0$, then L is abelian. We assume for now that $\dim[L, L] = 1$. Let e_1 be a basis element in $[L, L]$. We consider two cases.

If e_1 is in the center

$$Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\},$$

then we take $e_2, e_3 \in L \smallsetminus [L, L]$ such that $\{e_1, e_2, e_3\}$ is a basis of L. Then $[e_1, e_2] = 0$ and $[e_1, e_3] = 0$. How about $[e_2, e_3]$? We know that $[e_2, e_3]$ is an element of $[L, L]$, so we can write $[e_2, e_3] = \lambda e_1$. Since L is nonabelian, we have $\lambda \neq 0$, whence

$$z = e_1, \quad x = \frac{e_2}{\lambda}, \quad y = e_3$$

generates the Heisenberg algebra.

Suppose now that e_1 is not in $Z(L)$. Then we can find $\tilde{e}_2 \in L$ such that $[e_1, \tilde{e}_2] = \lambda e_1$ for some $\lambda \neq 0$. We set $e_2 = \frac{\tilde{e}}{\lambda}$, so that $[e_1, e_2] = e_1$. We choose $\tilde{e}_3$ such that $\{e_1, e_2, \tilde{e}_3\}$ is linearly independent and observe that

$$[e_1, \tilde{e}_3] = \mu e_1 \qquad \text{and} \qquad [e_2, \tilde{e}_3] = \nu e_1$$

for some scalars μ and ν. Setting,

$$e_3 = \tilde{e}_3 - \mu e_1 - \nu e_2,$$

we see that $\{e_1, e_2, e_3\}$ is a basis of L such that

$$[e_1, e_3] = [e_2, e_3] = 0.$$

It thus follows that

$$L = L^1 \oplus \mathbb{C}e_3,$$

where L^1 is a linear span of $\{e_1, e_2\}$ such that $[e_1, e_2] = e_1$.

We now consider L with $\dim[L, L] = 2$. We claim that $[L, L]$ is abelian. Suppose for a contradiction that $[L, L]$ is not abelian, and choose a basis $\{e_1, e_2\}$ of $[L, L]$. Since $[L, L]$ is a two-dimensional complex Lie algebra, we can assume by the previous classification result that $[e_1, e_2] = e_1$. Pick $e_3 \notin [L, L]$ and observe that

$$[e_3, e_1] = [e_3, [e_1, e_2]] = [e_1, [e_2, e_3]] - [e_2, [e_3, e_1]],$$

where the last identity follows from the Jacobi identity. Since both $[e_1, [e_2, e_3]]$ and $[e_2, [e_3, e_1]]$ belong to $[L, L]$, they are scalar multiples of e_1, and so

$$[e_3, e_1] = \mu e_1.$$

Another application of the Jacobi identity yields

$$\begin{aligned} 0 &= [e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] \\ &= [e_1, [e_2, e_3]] + [e_2, \mu e_1] + [e_3, e_1] \\ &= [e_1, [e_2, e_3]] - \mu e_1 + \mu e_1 \\ &= [e_1, [e_2, e_3]]. \end{aligned}$$

Since $[e_2, e_3] \in [L, L]$, we see that $[e_2, e_3]$ is a scalar multiple of e_1, whence $[L, L]$ is one-dimensional. Of course, this is absurd, and we conclude that $[L, L]$ is abelian.

We now fix $e_3 \notin [L, L]$. Recall that, for each $x \in L$,

$$\operatorname{ad} x(y) = [x, y]$$

is a linear endomorphism on $[L, L]$. Since L is not abelian, the operator $\operatorname{ad} e_3$ is nonzero, whence $\operatorname{ad} e_3$ has two eigenvalues, not necessarily distinct.

If $\operatorname{ad} e_3$ has two distinct eigenvalues λ_1 and λ_2, then we can find a basis $\{e_1, e_2\}$ in $[L, L]$ such that

$$[e_3, e_1] = \lambda_1 e_1 \qquad \text{and} \qquad [e_3, e_2] = \lambda_2 e_2;$$

in fact, we may choose $\lambda_1 = 1$ and $\lambda_2 = \lambda \neq 1$. This gives rise to an infinite family of Lie algebras.

If $\operatorname{ad} e_3$ has only one distinct eigenvalue, then we can take the Jordan canonical basis $\{e_1, e_2\}$ such that

$$\begin{aligned} [e_1, e_2] &= 0; \\ [e_1, e_3] &= e_1; \\ [e_2, e_3] &+ e_1 + e_2. \end{aligned}$$

Finally, we consider the case $\dim[L, L] = 3$. In this case, we have $[L, L] = L$. We claim that L is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. To see this, we let $\{e_1, e_2, e_3\}$ be a basis in L. Then $\{[e_i, e_j] : i < j\}$ is a linearly independent set, and so $[x, y] \neq 0$ for any linearly independent x and y in L.

Lemma 1.9. *For all $\theta \in L$, the operator $\operatorname{ad} \theta$ has a nonzero eigenvalue that is not nilpotent.*

Proof of lemma. Pick a nonzero element x of L. Then $\operatorname{ad} x$ has rank 2. If $\operatorname{ad} x$ is nilpotent, there exists Jordan basis x, y, z such that

$$\begin{aligned}[x,z] &= y;\\ [x,y] &= x;\\ [x,x] &= 0.\end{aligned}$$

Take $\theta = y$. Then x is an eigenvector of $\operatorname{ad}\theta$ with eigenvalue -1. □

Corollary 1.10. *Eigenvalues of* θ *are* 0, 1, *and* -1.

Proof of corollary. Since $\theta \in [L, L]$, the trace of $PQ - QP$ is zero. Therefore, the trace of θ is

$$0 = 0 + (-1) + 1,$$

as desired. □

It thus follows that

$$\begin{aligned}[\theta,x] &= -x;\\ [\theta,z] &= z;\\ [x,z] &= \theta.\end{aligned}$$

Exercise 1.11. Fininsh the above proof by explicitly finding the generators e, f, and h for $\mathfrak{sl}_2(\mathbb{C})$.

5. The general linear Lie algebra

Before we move on, let us consider a few more examples. Given a field k of characteristic zero, we define $\mathfrak{gl}_n(k)$, the Lie algebra of n-by-n matrices over k with the Lie bracket $[A, B] = AB - BA$. It is a well-known fact that

$$\mathfrak{gl}_n(k) = \mathfrak{sl}_n(k) \oplus k \cdot \mathbf{1},$$

where $\mathfrak{sl}_n(k)$ is the Lie subalgebra of $\mathfrak{gl}_n(k)$ consisting of matrices of trace zero.

Exercise 1.12. Prove that $\mathfrak{sl}_n(k)$ and $k \cdot \mathbf{1}$ are Lie ideals.

Similarly, we define the Lie subalgebra $\mathfrak{b}_+(n)$ of $\mathfrak{gl}_n(k)$, consisting of upper triangular matrices: the definition of $\mathfrak{b}_-(n)$ is clear. We also define the Lie subalgebra $\mathfrak{n}_+(n)$ of $\mathfrak{gl}_n(k)$, consisting of upper triangular matrices with zero diagonal entries; again, the definition of the analogus subalgebra $\mathfrak{n}_-(n)$ should be clear. $\mathfrak{h}(n)$ is the Lie algebra of diagonal matrices: this is an abelian subalgebra discovered by E. Cartan. The *orthogonal Lie algebra* σ_n consists of all n-by-n matrices such that $A^T = -A$.

Exercise 1.13. Check that the Lie subalgebras of $\mathfrak{gl}_n(k)$ defined above are, in fact, Lie subalgebras.

With this new array of examples in mind, we can recognize the complex Lie algebras of dimension 3.

Exercise 1.14. The complex Lie algebras of dimension 3 are $\mathfrak{sl}_2(k)$, $\mathfrak{n}_+(3)$ (which is the Heisenberg algebra), and $b_+(2)$.

Exercise 1.15. Show that $b_+(2)$ is isomorphic to a direct sum of a two-dimensional nonabelian complex Lie algebra and a one-dimensional complex Lie algebra.

CHAPTER 2

Representations of Lie Algebras

1. Definitions and examples

Definition. Let V be a vector space over k, and $\mathfrak{gl}(V)$ a Lie algebra of linear maps of V. A *representation* of a Lie algebra L on V is a Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$.

Here V is considered to be a module over L. For each $x \in L$, the map $\rho(x) : V \to V$ is a linear endomorphism on V. Given any $x, y \in L$, we have

$$\rho([x,y])v = \rho(x)(\rho(y)v) - \rho(y)(\rho(x)v)$$

for each $v \in V$.

Let us consider the basic examples. Of course, there is always the trivial representation, defined to be $\rho(x)v = 0$ for all $v \in V$. For each Lie algebra L, we can also define the *adjoint representation* as follows: we set $V = L$, $\rho = \mathrm{id}$, and

$$\rho(x)v = [x,v] = \operatorname{ad} x(v)$$

for each $x \in L$ and $v \in V$. If L is a general linear Lie algebra $\mathfrak{gl}_n(k)$ or any of its subalgebra, then we can define the *tautological representation* of L by setting $V = k^n$ and

$$\rho(x)v = x \cdot v$$

for each $x \in L$ and $v \in V$.

Definition. Given a representation ρ of a Lie algebra L, we define the *kernel* $\ker \rho = \{x \in L : \rho(x) =\}$. We say that ρ is *faithful* if $\ker \rho = \{0\}$.

We remark that the tautological representation of $\mathfrak{gl}_n$ is faithful.

Exercise 2.1. Study the adjoint representation for the general linear Lie algebra and its Lie subalgebras.

Exercise 2.2. The adjoint representation of L is faithful if and only if $Z(K) = \{0\}$.

Definition. The linear subspace V' of V is a *subrepresentation* if V' is a $\rho(x)$-invariant subspace for each $x \in L$.

If $L = \mathfrak{gl}_n$ with the adjoint representation, then $V' = \mathfrak{sl}_n$ is an example of a subrepresentation.

Definition. A representation $\rho : L \to \mathfrak{gl}(V)$ is *irreducible* if any ρ-invariant subspace of V is either V or $\{0\}$.

For the sake of example, we consider $L = \mathfrak{gl}_n$ with the tautological representation ρ. We define the direct sum

$$(V_1, \rho_1) \oplus (V_2, \rho_2) = (V, \rho)$$

by setting

$$\begin{aligned} V &= V_1 \oplus V_2 \\ \rho(x)(v_1 \oplus v_2) &= \rho_1(x)v_1 \oplus \rho_2(x)v_2. \end{aligned}$$

Definition. $\rho : L \to \mathfrak{gl}(L)$ is *completely reducible* if, for all $V' \subseteq V$, there exists a subspace $V'' \subseteq V$ such that

$$V \cong V' \oplus V''.$$

Proposition 2.3. *Any completely reducible finite-dimensional representation is isomorphic to a direct sum of irreducible representations.*

PROOF. Induction on $\dim V$. □

Exercise 2.4. Let ρ be the adjoint representation of the Lie algebra L. Then the following holds:

(1) The subrepresentations of ρ are the Lie ideals in L.
(2) L is simple if and only if ρ is irreducible.
(3) ρ is faithful if and only if $Z(L) = \{0\}$.

Proposition 2.5. $\mathfrak{gl}_n(k) = k \cdot 1 \oplus \mathfrak{sl}_n(k)$.

PROOF. It is enough to show that $\mathfrak{sl}_n(k)$ is irreducible representation of $\mathfrak{gl}_n(k)$. $\operatorname{ad}\mathfrak{gl}_n(k)$ acts on $\mathfrak{sl}_n(k)$, for $\mathfrak{sl}_n(k)$ is an ideal in $\mathfrak{gl}_n(k)$.

We shall only prove the proposition for $n = 2$. The case for $n > 2$ is left as an exercise. We let $\{e, h, f\}$ be the standard basis in $\mathfrak{gl}_2$. Let W be nonzero a $\mathfrak{gl}_2$-invariant subspace in $\mathfrak{sl}_2$.

We claim that $W = \mathfrak{sl}_2$. To see this, we pick a nonzero $x = ae + bh + cf$ in W. Then $[e, x] = -2be + ch \in W$ and $[e, [e, x]] = -2ce \in W$. If $c \neq 0$, then $e \in W$ and $h \in W$. If $c = 0$ and $b \neq 0$, then $e \in W$. If $c = 0$ and $b = 0$, then $a \neq 0$, and so $e \in W$. Similarly, $f \in W$, and so $[e, f] = h \in W$ as well. It follows that $W = \mathfrak{sl}_2$. □

Exercise 2.6. Complete the above proof for $n > 2$.

Exercise 2.7. The tautological representation of $\mathfrak{gl}_n(k)$ is irreducible.

Proposition 2.8. *The tautological representation of* $\mathfrak{n}_+(k)$ *is reducible but indecomposable.*

PROOF. Since

$$\mathfrak{n}_+ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0,$$

we see that

$$k \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0,$$

is an invariant subspace.

Consider the case $n = 2$. We have

$$\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and so $\mathfrak{n}_+(2)x \in k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It shows that we cannot find an invariant subspace W such that

$$k^2 = k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus W.$$

□

2. Invariant Bilinear Forms

Definition. Let L be a Lie algebra and $\rho : L \to \mathfrak{gl}(V)$ a representation of L. A bilinear form $\phi : V \times V \to k$ is *L-invariant* if

$$\phi(\rho(v_1), v_2) + \phi(v_1, \rho(v_2)) = 0.$$

Let us consider a few examples.

Exercise 2.9. If $L = SO_n$, $V = k^n$, and

$$\phi(v, w) = \sum_{i=1}^{n} v_i w_i,$$

then ϕ is L-invariant.

Take the adjoint representation of $\mathfrak{gl}_n$, and define

$$\phi(A, B) = \operatorname{tr}(AB).$$

Then ϕ is $\mathfrak{gl}_n$-invariant. In fact,

$$\phi([x, A], B) + \phi(A, [x, B]) = \operatorname{tr}([x, A]B + A[x, B]) = \operatorname{tr}([x, AB]) = 0.$$

Exercise 2.10. If $V = \mathrm{Mat}_n(k)$ and $\phi(A, B) = \operatorname{tr} A(B^t)$, then ϕ is not $\mathfrak{sl}_n$-invariant, but SO_n-invariant.

We now consider an important example known as the *Killing form*, or the *Cartan-Killing form*. The following is a mandatory exercise:

Exercise 2.11. Take the adjoint representation of a Lie algebra L and define

$$K(x, y) = \operatorname{tr}(\operatorname{ad} x \cdot \operatorname{ad} y).$$

Then K is L-invariant.

The first application of invariant forms is as follows:

Proposition 2.12. *Let (V, ρ) be a representation and $\phi : V \times V \to k$ an invariant bilinear form. If W is a subrepresentation, then*

$$W^{\perp} = \{v \in V : \phi(w, v) = 0 \text{ for all } w \in W\}.$$

is also a subrepresentation.

Proof. Let $v \in W^{\perp}$. We wish to check that $\rho(x)v \in W^{\perp}$ for all $x \in L$. Then

$$\phi(\rho(x)v, w) = -\phi(v, \rho(x)w) = 0,$$

and so $\rho(x)v \in W^{\perp}$. □

Corollary 2.13. $\ker \phi = V^{\perp}$ *is a subrepresentation, and so the kernel* $\ker K$ *of the Killing form* K *is an ideal in* $L = V$*.*

Corollary 2.14. *Let* $k = \mathbb{R}$*. Assume that* ϕ *is invariant and positively defined, i.e.,* $\phi(v, v) > 0$*. If* W *is not a trivial subrepresentation, then*

$$V = W \oplus W^{\perp}.$$

3. Homomorphisms of Representations

Definition. Given a pair (V_1, ρ_1) and (V_2, ρ_2) of representations of L, we say that a linear map $\phi : V_1 \to V_2$ is a *homomorphism of representations*, or an *intertwining operator* if

$$\begin{tikzcd}
V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(x)"'] & V_2 \arrow[d, "\rho_2(x)"] \\
V_1 \arrow[r, "\varphi"'] & V_2
\end{tikzcd}$$

is commutative for all $x \in L$. The linear space of all homomorphisms (V_1, ρ_1) to (V_2, ρ_2) is denoted by $\mathrm{Hom}_L(V_1, V_2)$.

Theorem 2.15 (Schur's lemma)**.** *If* $\varphi : V_1 \to V_2$ *a homomorphism of irreducible representations, then* φ *is either trivial or an isomorphism.*

PROOF. $\ker \varphi$ is a subrepresentation of V_1. Since V_1 is irreducible, $\ker \varphi$ is either 0 or V_1. The former implies that $\varphi = 0$. The latter shows that φ is injective. In this case, $\mathrm{im}\, \varphi$ is nontrivial, whence by the irreducibility of V_2 we have $\mathrm{im}\, \varphi = V_2$. It follows that φ is an isomorphism. ☐

Corollary 2.16. *Any endomorphism of a finite-dimensional irreducible complex representation* (V, ρ) *is a multiplication-by-a-scalar map.*

PROOF. Any endomorphism $\varphi : (V, \rho) \to (V, \rho)$ has an eigenvalue λ. Then $\varphi - \lambda \cdot \mathrm{id}$ is a homomorphism with a nontrivial kernel, whence by Schur's lemma $\varphi = \lambda \cdot \mathrm{id}$. ☐

Exercise 2.17. Let (V, ρ) be a representation and V a subrepresentation. Then there exists a subrepresentation $V'' \subseteq V$ such that

$$V = V' \oplus V''$$

if and only if there exists a homomorphism $p : V \to V$, called a *projection*, such that $\mathrm{im}\, p = V'$ and $p^2 = p$

4. Tensor Products and Duality

Definition. Let (V_1, ρ_1) and (V_2, ρ_2) be representations of the Lie algebra L. The *tensor product* of (V_1, ρ_1) and (V_2, ρ_2) is a representation (V, ρ) such that $V = V_1 \otimes V_2$ and that the Liebniz rule

$$\rho(x)(v_1 \otimes v_2) = \rho_1(x)v_1 \otimes v_2 + v_1 \otimes \rho_2(x)v_2.$$

Exercise 2.18. The tensor product is well-defined, viz.,

$$\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x, y]).$$

Note that we can also define

$$\bigotimes_{i=1}^{n}(V_i, \rho_i),$$

and show that the space S^nV of symmetric tensors and the space $\wedge^nV$ of antisymmetric tensors are $V^{\otimes n}$-invariant subspaces.

Definition. Let (V, ρ) be a representation of a Lie algebra L. The *dual representation*, or the *contragredient*, (V^*, ρ^*) of (V, ρ) is a map $\rho^* : L \to \mathfrak{gl}(V^*)$ such that

$$[\rho^*(x)f](v) = -f(\rho(x)v)$$

for all $x \in L$, $f \in V^*$, and ρ^*.

Exercise 2.19. Check that ρ^* is a representation.

Let V and W be vector spaces. Then

$$\operatorname{Hom}(V, W) \cong V^* \otimes W.$$

Let $f \in V^*$. Then

$$(f \otimes w)(v)f(v)w.$$

Bilinear forms $V \times V \to k \cong \operatorname{Hom}(V \otimes V, k) \cong V^* \otimes V^* \otimes k \cong V^* \otimes V^*$ If $\dim V$ is finite, then $(V \otimes V)^* = V^* \otimes V^*$.

Proposition 2.20. *Let V be a tautological representations of $\mathfrak{gl}_n(k)$. Then adjoint representations of $\mathfrak{gl}_n(k)$ is isomorphic to $V^* \otimes V$ (tensor product as representations).* ☐

Definition. Let L be a Lie algebra, and ρ a representation of L. We define the space

$$V^L = \{v \in V : \rho(x)v = 0 \text{ for all } x \in L\}$$

of invariants.

Proposition 2.21. $\operatorname{Hom}_L(V, W) \cong (V^* \otimes W)^L$. ☐

Proposition 2.22. *The space of invariant bilinear forms on V is isomorphic to $(V^* \otimes V^*)^L$.* ☐

5. Represenations of $\mathfrak{sl}_2$

Let $V \cong k^2$ be tautological representation of $\mathfrak{sl}_2(k)$, where k is either $\mathbb{R}$ or $\mathbb{C}$. We describe S^nV. The basis is given by

$$e_1 \otimes \cdots e_1 \otimes e_2 \otimes \cdots \otimes e_2 = e_1^k e_2^{n-k}.$$

(there are k e_1 and $n - k$ e_2)

Claim. $S^nV \cong k_n[u, v]$, where the isomorphism is given by

$$e_1^k e_2^{n-k} \mapsto u^k v^{n-k}.$$

$\mathfrak{sl}_2$ has canonical generators

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They act on S^nV. How do they act on $k_n[u,v]$?

Here is the answer, left as an exercise:

Exercise 2.23. The action of $\mathfrak{sl}_2(k)$ on S^nV is given by the map $\rho : \mathfrak{sl}_2(k) \to \mathfrak{gl}(k_n[u,v])$, which is defined by

$$\rho(e) = u\frac{\partial}{\partial v},$$

$$\rho(f) = v\frac{\partial}{\partial u},$$

$$\rho(h) = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Proposition 2.24. *ρ is a representation of $\mathfrak{sl}_2(k)$ on $k_n[u,v]$.* □

We have to check

$$\begin{aligned} \rho(e)\rho(f) - \rho(f)\rho(e) &= \rho(h) \\ \rho(h)\rho(e) - \rho(e)\rho(h) &= 2\rho(e) \\ \rho(h)\rho(f) - \rho(f)\rho(h) &= -2\rho(f). \end{aligned}$$

Similarly, we let $\frac{\partial}{\partial u}$. Multiplication by u acts on $k[u]$. What is $[\frac{\partial}{\partial u}, u] = 1$.

Theorem 2.25. *The representation S^nV is irreducible.*

PROOF. We show that $S^nV \cong k_n[u,v]$.

Claim. Any nontrivial invariant subspace in $k_n[u,v]$ contains u^n.

In fact, we let

$$w = \alpha_0 u^n + \alpha_1 u^{n-1}v + \cdots + \alpha_n v^n \in W.$$

Let α_k be nonzero scalar with maximum k. Then

$$w = \alpha_k u^{n-k}v^k + \alpha_{k+1}u^{n-k+1}v^{k-1} + \cdots$$

We apply $\rho(e)^k w = \alpha_k k! u^n$. So, $u^n \in W$.

Claim. $k_n[u,v]$ is generated by u^n.

In fact,

$$\begin{aligned} \rho(f)^k u^n &= \rho(f)^k u^n \\ &= v^k \left(\frac{\partial}{\partial u}\right)^k u^n \\ &= n(n-1)\cdots(n-k+1)v^k u^{n-k}. \end{aligned}$$

and so $v^k u^{n-k} \in W$. So $W = k_n[u,v]$. It proves that S^nV is irreducible. □

Note that

$$\begin{aligned} \rho(e)u^n &= 0 \\ \rho(h)u^n &= n \cdot u^n. \end{aligned}$$

Also, S^nV is the "highest weight" module, where the highest weight equals n. Here, the highest meansa the maximum of all eigenvalues of $\rho(h)$.

Any irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension n is isomorphic to S^nV.

CHAPTER 3

Structure Theory of Lie Algebras

1. Universal enveloping algebras

Recall that any associative algebra A has a Lie algebra structure $[a,b]=ab-ba$. Let L be a Lie algebra, and A an associative algebra. A Lie homorphism $\psi : L \to A$ means that we have

$$\psi([x,y]) = \psi(x)\psi(y) - \psi(y)\psi(x)$$

for all $x, y \in L$.

Definition. Let L be a Lie algebra over k. A *universal enveloping algebra* of L is a pair $(U(L), \varepsilon)$, where $U(L)$ is an associative algebra with 1, $\varepsilon : L \to U(L)$ is a Lie homomorphism, and, for any associative algebra A and a Lie homomorphism $\varphi : L \to A$, there exists a unique homomorphism $\tilde{\varphi} : U(L) \to A$ as associative algebras such that $\varphi = \tilde{\varphi} \circ \varepsilon$. That is, the following diagram commutes:

$$\begin{tikzcd}
L \arrow[r, "\varepsilon"] \arrow[dr, "\varphi"'] & U(L) \arrow[d, "\tilde{\varphi}"] \\
 & A
\end{tikzcd}$$

For any representation $\rho : L \to \mathfrak{gl}(V)$, V is a $U(L)$-module. Why is this true? From the map $\rho : L \to \mathfrak{gl}(V)$, we get $\varphi : L \to \mathrm{End}(V)$. $\mathrm{End}(V)$ is an associative algebra, and so we obtain $\tilde{\varphi} : U(L) \to \mathrm{End}(V)$. This makes V a $U(L)$-module.

Proposition 3.1. *$U(L)$ is unique up to an isomorphism.*

Proof. Let $U_1(L)$ and $U_2(L)$ be universal enveloping algebras of L, and $\varepsilon_i : L \to U_i(L)$ the corresponding maps. The proposition now follows from the universal property. ☐

Theorem 3.2. *$(U(K), \varepsilon)$ exists for any Lie algebra L.*

Proof. We define $U(L)$ by generators and relations. First, we construct the tensor algebra (free algebra)

$$T(L) = k \oplus L \oplus L \otimes L \oplus L \otimes L \otimes L \oplus \cdots$$

Here we treat L as a vector space. we set

$$(w_1 \otimes \cdots \otimes w_k) \cdot (w'_1 \otimes \cdots \otimes w'_m) = w_1 \otimes \cdots \otimes w_k \otimes w'_1 \otimes \cdots \otimes w'_m.$$

Let $J \subseteq T(L)$ be two-sided algebra generated by $x \otimes y - y \otimes x$, $x, y \in L$. Set $U(L) = T(L)/J$ and $L \hookrightarrow T(L) \to U(L)$.

We now prove the universality. Let A be an associative algebra with 1, and $\varphi : L \to A$ a Lie homomorphism. Let $x_1, x_2, \ldots$ be a linear basis in L. We construct

$\pi(x_i) = \varphi(x_i)$ such that

$$\begin{tikzcd}
L \arrow[r] \arrow[dr, "\varphi"'] & T(L) \arrow[d, "\psi"] \\
 & A
\end{tikzcd}$$

Note that $T(L)$ is generated by x_i. $\psi(1) = 1$. Therefore, ψ defines a homomorphism of associative algebras $T(L) \to A$. Note that

$$\psi(x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]) = 0.$$

It follows that $\psi(J) = 0$, and so

$$\begin{tikzcd}
L \arrow[r] \arrow[dr] & T(L) \arrow[r] \arrow[d, "\psi"'] & U(L) \arrow[dl, "\tilde{\psi}"'] \\
 & A &
\end{tikzcd}$$

□

Example. If L is an abelian Lie algebra with a basis $x_1, \dots, x_n$, then

$$U(L) = T(L)/ < x_i \otimes x_j - x_j \otimes x_i >= S(L)$$

is a commutative algebra with variables $x_1, \dots, x_n$.

Let L be a Lie algebra with a basis $x_1, \dots, x_n$ and the bracket defined by

$$[x_i, x_j] = \sum c_{ij}^k x_k,$$

where c_{ij}^k is the structural constant. We recall that the universal enveloping algebra $U(L)$ of L is generated by $x_1, \dots, x_n$ and the relation

$$x_i x_j - x_j x_i - \sum_{k=1}^{n} c_{ij}^k x_k,$$

where $x_i = \varepsilon(x_i)$. We say that $U(L)$ is a quadratic-linear algebra[1].

We now define the coproduct. We let A be a vector space over k. The *coproduct* is $\Delta : A \to A \otimes A$ such that

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, equal] & A \otimes A \arrow[r, "\Delta \circ \mathrm{id}"] & (A \otimes A) \otimes A \arrow[d, equal] \\
A \arrow[r, "\Delta"'] & A \otimes A \arrow[r, "\mathrm{id} \circ \Delta"'] & A \otimes (A \otimes A)
\end{tikzcd}$$

commutes. This property is called *coassociativity.* Note that we have

$$A^* \otimes A^* \xrightarrow{\Delta^*} A^*,$$

so that A^* is an associative algebra.

Definition. A is *bialgebra*, or *bigebra*, if A is an associative algebra and Δ is an algebra homomorphism.

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[1] The theory of such algebras can be found in Poleschuk and Positselski, *Quadratic Algebras.*

Example. For $U(L)$, we have $\Delta : U(L) \to U(L) \otimes U(L)$ defined by

$$\Delta(\varepsilon(x)) = \varepsilon(x) \otimes 1 + 1 \otimes \varepsilon(x)$$

for each $x \in L$. We extend this to an algebra homomorphism.

Proposition 3.3. *Δ is a coproduct.*

Sketch of proof. It is enough to check

$$\Delta(x_i)\Delta(x_j) - \Delta(x_j)\Delta(x_i) = \Delta([x_i, x_j])$$

(why?). Here $x = \varepsilon(x)$, $x_1, \dots, x_n$ basis in L. The left-hand side equals

$$x_i x_j \otimes 1 + x_i \otimes x_j + x_j \otimes x_i + 1 \otimes x_i x_j$$

(from $(x_i \otimes 1 + 1 \otimes x_i)(x_j \otimes 1 + 1 \otimes x_j)$)

$$-1 \otimes x_i x_j - x_i \otimes x_j - x_j \otimes x_i - x_i x_j \otimes 1.$$

This whole thing equals

$$= [x_i, x_j] \otimes 1 - 1 \otimes [x_i, x_j],$$

which equals the right-hand side. □

Exercise 3.4. Δ defined above is cocommutative. *Hint: Show that Δ^* is commutative.*

Exercise 3.5. Consider the group algebra $k[G]$ with $\Delta(g) = g \otimes g$, and find the identity (1) for $U(L)^*$ and $k[G]^*$.

2. The Poincaré-Birkhoff-Witt Theorem

We now discuss the Poincaré-Birkhoff-Witt theorem.

Definition. An algebra A over k is *graded* if

$$A = \bigoplus_{n \geq 0} A_n$$

such that $A_n \cdot A_m \subseteq A_{n+m}$.

Example. $A = k[x_1, \dots, x_n]$, $A_0 = k$,

$$A_m = \operatorname{span}\{x_1^{i_1}, \dots, x_n^{i_n} : i_1 + \cdots + i_n = m\}.$$

In particular, $A_1 = \operatorname{span}\{x_1, \dots, x_n\}$.

Example. Let $A = k\langle x_1, \dots, x_n\rangle$, the free associative algebra generated by $x_1, \dots,$ x_n, or $A = T(V)$, the tensor algebra

$$\bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

over $V = \operatorname{span}\{x_1, \dots, x_n\}$. (in fact, these two are the same).

Example. Let $A = T(V)/I$, where I is a two-sided *homogeneous* ideal, viz., I is generated by $f_i \in A_{n_i}$ for $i = 1, \dots, k$. For example, we can pick an ideal I generated by $\{x_p x_q - x_q x_p\}$ in $T(V)_2$. In this case, $T(V)/I = k[x_1, \dots, x_n]$.

Is $U(L)$ graded? No, the relations are $x_i x_j - x_j x_i - [x_i, x_j]$.

Definition. An associative algebra B over k is *filtered* if there exists a sequence of vector spaces

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

such that

$$\bigcup_{n \geq 0} F_n = B$$

and $F_n \cdot F_m \subseteq F_{n+m}$.

Example. $A = \bigoplus_{n \geq 0} A_n$, a graded algebra. We let

$$F_i = A_0 \oplus A_1 \oplus \cdots \oplus A_i.$$

Example. $U(L)$. We set $F_0 = k$, $F_1 = L$, F_2 generated by $x_i x_j$ and x_k, and, in general, F_k generated by all monomials of common degree at most k. Note that $x_i x_j - x_j x_i$ in $U(L)$ is also in $F_1 \cdot F_1$. We have $F_1 \cdot F_1 \subseteq F_2$.

Exercise 3.6. $F_n \cdot F_m \subseteq F_{n+m}$.

To every filtered algebra B with $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ corresponds a *graded algebra*

$$\operatorname{gr} B = \bigoplus_{n \geq 0} F_n / F_{n-1},$$

where $F_{-1} = \{0\}$. What is the algebra structure in $\operatorname{gr} B$? Let $f \in F_n/F_{n-1}$, $g \in F_m/F_{m-1}$. What is $fg =$? We let $\tilde{f}$ and $\tilde{g}$ be representations of f and g in F_n and F_m, respectively. Then $\tilde{f}\tilde{g} = \tilde{h}$ is in F_{n+m}. Take image of $\tilde{h}$ in F_{n+m}/F_{n+m-1}.

Set $fg = h$. Why does h not depend on the choice of $\tilde{f}\tilde{g}$? Choose any representative $\tilde{\tilde{f}}$ and $\tilde{\tilde{g}}$.

$$\tilde{\tilde{f}} = \tilde{f} + f_0,$$

where $f_0 \in F_{m-1}$, and

$$\tilde{\tilde{g}} = \tilde{g} + g_0,$$

where $g_0 \in F_{n-1}$. Then $\tilde{\tilde{f}}\tilde{\tilde{g}} - \tilde{f}\tilde{g} \in F_{m+n-1}$, so h is defined uniquely.

Proposition 3.7. *If L is a Lie algebra, then $\operatorname{gr} U(L)$ is a commutative algebra generated by L.*

Proof. Let $x, y \in L$. Take $xy - yx \in L \subseteq U(L)$ in F_2/F_1, the image of $xy - yx$ is zero. so $\dot{x}\dot{y} = \dot{y}\dot{x}$, where $\dot{x}$ is the projection of x in F_2/F_1, and $\dot{y}$ the projection of y in F_2/F_1. So $\dot{x}_1, \ldots, \dot{x}_n$ in $\operatorname{gr} U(L)$ commute. Here $x_1, \ldots, x_n$ is a basis in L and $\operatorname{gr} U(L)$ is commutative. In other words, each element in $U(L)$ equals $\alpha x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ + sum of monomials of lesser degrees, where $\alpha \in k$. That is to say, given $x_{i_1}, \ldots, x_{i_m}$, we reorder to have $i_1 \leq i_2 \leq \cdots \leq i_m$. It follows that we have the homomorphism $S(L) \to \operatorname{gr} U(L)$, where $S(L) = k[x_1, \ldots, x_n]$. ☐

Theorem 3.8 (Poincare-Birkhoff-Witt)**.** *$S(L) \cong \operatorname{gr} U(L)$, when $\operatorname{char} k = 0$.*

Proof. We have $S(L) \to \operatorname{gr} U(L)$, which is surjective. Is it a monomorphism? In other words, why are monomials linearly independent? We make use of the following

Lemma 3.9. *If PBW is true for L, then it is true for L/I, where I is a Lie ideal.*

Therefore, it is enough to prove the theorem for free Lie algebras. Recall that if X is a set, then the *free Lie algebra* L_X over X can be constructed such that there is an embedding $e : X \hookrightarrow L_X$ and, for any map $\varphi : X \to L$, there exists a unique Lie homomorphism $\tilde{\varphi}$ such that $\tilde{\varphi}e = \varphi$, viz.,

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "e"'] & L \\
L_X \arrow[ur, dashed, "\exists!\tilde{\varphi}"'] &
\end{tikzcd}$$

commutes. Furthermore, for $k\langle x_1, \ldots, x_n\rangle$, we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & A \\
k\langle\cdots\rangle \arrow[ur] &
\end{tikzcd}$$

and we have[2]

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\varepsilon"'] & A \\
U(L) \arrow[ur, dashed] &
\end{tikzcd}$$

In our case, $X = \{x_1, \ldots, x_n\}$.

If L has basis $x_1, \ldots, x_n$, then L_X maps surjectively onto L, and so $U(L_X) \cong k\langle x_1, \ldots, x_n\rangle$. $\square$

Exercise 3.10. If L_X exists, then it is unique up to an isomorphism.

We now construct L_X. Let $V = k \cdot X$, $T(V) = k\langle X\rangle$, the free associative algebra defined by X and L_X linear subspace in $T(V)$ generated by $x \in X$ and operation $[a, b] = ab - ba$. We get a Lie algebra this way because $(T(V), [\cdot, \cdot])$ is a Lie algebra.

Note that L_X is graded: we have the decomposition

$$L_X = \bigoplus_{i \geq 1} L_i$$

such that $[L_i, L_j] \subseteq L_{i+j}$; here $L_1 = V_1$, $L_2 = \operatorname{span}_{i<j}([x_i, x_j])$, where $x = \{x_1, \ldots, x_n\}$.

L_X is spanned by elements

$$[x_{i_k}, [x_{i_{k-1}}, \ldots, [x_{i_2}, x_{i_1}], \ldots].$$

Note that this is not a basis. Constructing a basis is a combinatorial problem, which we will not pursue in this course.

We also remark that $T(V)$ has a coproduct $\Delta : T(V) \to T(V) \otimes T(V)$.

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{m=0}^{n} (x_1 \otimes \cdots x_m) \otimes (x_{m+1} \otimes \cdots \otimes x_n).$$

If $m = 0$, then $1 \otimes x_1 \otimes \cdots \otimes x_n$. If $m = n$, then $(x_1 \otimes \cdots \otimes x_n) \otimes 1$.

Theorem 3.11. *If* $\operatorname{char} k = 0$, *then* $L_X = \{w \in T(V) : \Delta W = 1 \otimes w + w \otimes 1\}$. $\square$

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[2] For more information, see Rentenauer, *Free Lie Algebras.*

We encourage the readers to check this theorem for $w = x \otimes y - y \otimes x$.

Exercise 3.12. $UL_X = T(V)$.

We want to prove the PBW theorem: if L is a Lie algebra, then $\operatorname{gr} U(L) \cong S(L)$.

Lemma 3.13. *If PBW is true for L, then it is true for L/I for any Lie ideal I.*

Proof of lemma. Let $x_1, \dots, x_n$ basis in L such that $x_m, \dots, x_n$ basis in I. Then $U(L) \cdot I \cdot U(L) \subseteq U(L)$ is the space spanned by $x_1^{k_1} \cdots x_n^{k_n}$, where $(k_m, \dots, k_n) \neq 0$. If PBW holds for L, then $\{x_1^{k_1}, \dots, x_n^{k_n}\}$ is a basis in $\operatorname{gr} U(L)$.

Note that $\operatorname{gr} U(L/I) = \operatorname{gr}[U(L)/U(L) \cdot IU(L)]$, and so a basis in $\operatorname{gr} U(L/I)$ is given by $x_1^{k_1} \dots x_{m-1}^{k_{m-1}}$. Then PBW holds for $U(L/I)$. □

Therefore, it is enough to prove PBW for free Lie algebras. We denote the free Lie algebra over $x_1, \dots, x_n$ by F_n.

Lemma 3.14. *If $\varepsilon : F_n \to U(F_n)$ is injective, then PBW is true for F_n.*

Proof of lemma. We proceed by induction on degree of $x_1^{k_1} \cdots x_n^{k_n}$. ($\deg = k_1 + \cdots + k_n$). If $\deg = 1$, then the elements $x_1, \dots, x_n$ are linearly independent.

We assume that the monomials $x_1^{k_1} \cdots x_n^{k_n}$ are linearly independent for $k_1 + \cdots + k_n \leq M$. We suppose also that there is a nontrivial linear combination of monomials $x_1^{k_1} \cdots x_n^{k_n}$ such that $k_1 + \cdots + k_n = M + 1$.

Since $\Delta : U(L) \to U(L) \otimes U(L)$, the linear combination of

$$\Delta(x_1^{k_1} \otimes \cdots \otimes x_n^{k_n}) - x_1^{k_1} \otimes \cdots \otimes x_n^{k_n}) \otimes 1 - 1 \otimes (x_1^{k_1} \otimes \cdots \otimes x_n^{k_n}).$$

for $k_1 + \cdots + k_n = M + 1$ is zero.

We now consider the sum

$$\sum_{l_i=1}^{k_i-1} c(l_1, \dots, l_n) x_1^{l_1} \cdots x_n^{l_n} \otimes x_1^{k_1 - l_1} \cdots x_n^{m_n - l_n}.$$

Since $x_1^{l_1} \cdots x_n^{l_n}$ are linearly independent and $x_1^{k_1-l_1} \cdots x_n^{m_n-l_n}$, all coefficients c are zero. It follows that there are no nontrivial linear combinations of $x_1^{k_1} \cdots x_n^{k_n}$. □

Lemma 3.15. $U(F_n) = k\langle x_1, \dots, x_n \rangle$. □

Lemma 3.16. *$\varepsilon : F_n \to U(F_n)$ is injective.* □

A proof of PBW can be found in Bourbaki, *Seminar "Sophus Lie"*, and another can be found in Jacobson, *Lie Algebras*.

3. Nilpotent Lie Algebras

Let L be a Lie algebra over k, and V and W be subspaces in L. Let

$$[V, W] = \operatorname{span}\{[a, b] : a \in V, b \in W\}.$$

Exercise 3.17. If V and W are Lie ideals, then $[V, W]$ is a Lie ideal.

With a Lie algebra L, we associate a central series by setting $L_{(1)} = L$, $L_{(2)} = [L, L]$, $L_{(3)} = [L, [L, L]]$, and so on:

$$L_{(1)} \supseteq L_{(2)} \supseteq L_{(3)} \supseteq \cdots$$

Exercise 3.18. $[L_{(i)}, L_{(j)}] \subseteq L_{(i+j)}$.

Exercise 3.19. The following conditions are equivalent:

(i) there exists n such that $L_{(n)} = 0$.

(ii) there exists n such that

$$[y_n, [y_{n-1}, \dots, [y_2, y_1], \dots] = 0$$

for all $y_1, y_2, \dots, y_n \in L$.

(iii) there exists a chain of Lie ideals $I_1, \dots, I_n$ such that

$$L \supseteq I_1 \supseteq I_2 \supseteq \dots \supseteq I_n = \{0\},$$

$[L, I_k] = I_{k+1}$ for all $k = 1, \dots, n-1$.

Hint: (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(i).*

Example. Abelian Lie algebras.

Example. $\mathfrak{n}(k)$. $\mathfrak{n}(3)$ is the Heisenberg algebra H. $[H, H] = ke_3$. $[H, [H, H]] = 0$.

Example. We consider a generalization of $\mathfrak{n}_+$. Let V be a vector space. Let a *flag* $\mathscr{F}$ of V be the chain

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_n = V$$

of vector spaces, where $\dim V_i = i$. Define

$$\mathfrak{n}(\mathscr{F}) = \{u \in \operatorname{End} V : uV_i \subseteq V_{i-1}, i = 1, \dots, n\}.$$

The claim is that $\mathfrak{n}(\mathscr{F})$ is nilpotent Lie subalgebra in $\mathfrak{gl}(V)$. Choose basis $e_1, e_2, \dots,$ e_n in V such that $\{e_1, \dots, e_i\}$ is a basis in V_i.

Exercise 3.20. $\mathfrak{n}(\mathscr{F})$ is the collection of strictly upper-triangular matrices in the basis $\{e_1, \dots, e_n\}$.

Exercise 3.21. Set $\mathfrak{n}_j(\mathscr{F}) = \{u : uV_i \subseteq V_{i-j}\}$. Then $[\mathfrak{n}(\mathscr{F}), \mathfrak{n}_j(\mathscr{F})] \subseteq \mathfrak{n}_j(\mathscr{F}) \subseteq \mathfrak{n}_{j+1}(\mathscr{F})$. Furthermore, $\mathfrak{n}_j(\mathscr{F}) = 0$ for $j >> 0$.

Example. Examples of non-nilpotent Lie algebras:

(i) $\mathfrak{sl}_n$: $[\mathfrak{sl}_n, \mathfrak{sl}_n] = \mathfrak{sl}_n$. We checked for $n = 2$.

(ii) If L is simple, then $[L, L] = L$.

Theorem 3.22. *Lie algebra L is nilpotent if and only if* $\operatorname{ad} x$ *is nilpotent for any L*

Proof. Assume, for all $x \in L$, that there exists $n_x \in \mathbb{N}$ such that $(\operatorname{ad} x)^{n_x} = 0$. Then there exists $n \in \mathbb{N}$ such that $(\operatorname{ad} x)^n = 0$. ☐

Theorem 3.23 (Engel)**.** *Let $\rho : L \to \mathfrak{gl}(L)$, $\rho(x)$ is nilpotent for any $x \in L$. Then there exists a flag*

$$\mathscr{F} : 0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V$$

such that $\rho(L) \subseteq \mathfrak{n}(\mathscr{F})$.

Each x has a flag $\mathscr{F}_x$.

Theorem 3.24. *There exists a common flag.*

Theorem 3.25 (Engel')**.** *Under the assumption of Engel's theorem, there exists $v \in V$ such that $v \neq 0$ but $\rho(x)v = 0$ for all $x \in L$.*

We show that the above two theorems are equivalent. Engel $\Rightarrow$ Engel' is clear, and Engel' $\Rightarrow$ Engel is induction on $\dim V$. For $\dim V = 1$, it is trivial. Given V, taken $\bar{V} = V/kv$ (where Engel is true). Flag

$$\bar{V}_1 \subseteq \bar{V}_2 \subseteq \cdots$$

$\rho(L)\bar{V}_i \subseteq \bar{V}_{i-1}$. Let V_{i+1} be a preimage of $\bar{V}_i$ for all i. Then $\mathscr{F}$

$$0 = V_0 \subseteq V_1 = kv \subseteq V_2 \subseteq V_3 \subseteq \cdots$$

satisfies Engel'.

We now prove Engel's theorem. In doing so, we shall remind ourselves that, in dealing with Lie algebras, there are three methods of proofs: induction, boring induction, and very boring induction.

Proof of Engel's theorem. We proceed by induction. Assume $L \subseteq \mathfrak{gl}(V)$. Note that $\operatorname{ad} x$ is a nilpotent operator on L (why?). Assume the theorem is true for nilpotent L', $\dim L' < \dim L$. Define

$$N(L') = \{x \in L : [x, L'] \subseteq L\},$$

the centralizer of L' in L. $N(L')$ is a Lie algebra.

Claim. $L'_1 \subsetneq N(L')$.

In fact, L' acts on L' and on L/L' (why?) The actions are nilpotent. By induction, there exists $\bar{x} \in L/L'$ such that $L'(\bar{x}) = 0$. Choose a representative of $\bar{x}$: $\bar{x} = x + L'$. Then for any $y \in L'$, $[y, x] \in L'$, so $[x, y] = -[y, x] \in L'$. Then $x \in N(L')$, $x \notin L'$. This proves the claim.

Claim. We also claim that for a nontrivial nilpotent Lie algebra L, there exists an ideal $L' \subseteq L$ such that $\operatorname{codim} L' = 1$.

In fact, let L' be maximum Lie subalgebra in L, $L' \neq L$. Such subalgebras exist, because $[L, L] \neq L$. Then $N(L') = L$ by Claim 1, so L' is ideal in L.

Take any 1-dimensional subpsace in L/L'; its preimage in L contains L' and is not equal to L'. Preimage is a Lie algebra, so the preimage is L. So, $\dim L/L' = 1$. This proves the claim.

Claim. If we let $W = \{v \in V : L'v = 0\}$, then W is L-invariant.

In fact, let $x \in L$, $y \in L'$, $w \in W$ $y(xw) = x(yw) - [x, y]v$. so, $[x, y]v = 0$. Therefore, $y(xw) = x(yw) = 0$ and so $xw \in W$. Therefore, W is L-invariant.

We now finish the proof of the theorem. We let $z \in L \smallsetminus L'$, $z \neq 0$ by Claim 2. z acts as nilpotent operator on W. Take $v \in W$, $v \neq 0$ and $zv = 0$. We have $zv = 0$, $L'v = 0$, $L = kz \oplus LL'$. It follows that $Lv = 0$. □

Corollary 3.26. *Lie algebra L' is nilpotent iff $\operatorname{ad} x$ is nilpotent for all $X \in L$.*

Proof. Suppose that L is nilpotent. $\operatorname{ad} x$ is nilpotent for all $x \in L$. By Engel's theorem, there exists

$$F_1(0) \subseteq L_1 \subseteq L_2 \subseteq L_n$$

such that $[L_i, L_i] = L_{i-1}$ (why?). Look at (1), (2) and (3) in the definition of L. □

Here is a group analog:

Definition. Let V be a finite-dimensional vector space over k. An element $g \in GL(V)$ is *unipotent* if one of the three equivalent relations holds

(i) $g = 1 + n$, where n is nilpotent and $1 = \mathrm{id}$.
(ii) There is a basis in V such that

$$g = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

(iii) All eigenvalues of g are equal to 1.

Exercise 3.27. Prove that the above three conditions are equivalent.

Theorem 3.28 (Kolchin)**.** *Let G be a subgroup of $GL(V)$ such that each $g \in G$ is unipotent. Then there exists*

$$F : \{0\} = V_0 \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

such that $gV_i \subseteq V_i$.

Since this is not a course on group theory, we shall not prove this theorem.

4. Solvable Lie Algebras

We first remark that there are non-nilpotent solvable Lie algebras.

Example. $\mathfrak{b} \subseteq \mathfrak{gl}_n$.

Given a flag

$$\mathscr{F} : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V,$$

we defined a Lie algebra

$$\mathfrak{b}(\mathscr{F}) = \{u \in \operatorname{End} V : uV_i \subseteq V_i\}.$$

Let L be a Lie algebra. If we set $D^1L = L$, $D^2L = [L, L]$, then $D^kL = [D^{k-1}L, D^{k-1}L]$, and so

$$D^3L = [[L, L], [L, L]].$$

Therefore,

$$L = D^1L \supseteq D^2L \supseteq D^3L \supseteq \cdots .$$

which is referred to as the *derived series.*

Definition. A Lie algebra L is *solvable* if it admits a finite derived series.

Exercise 3.29. Find the derived series for $\mathfrak{b}_n$.

Exercise 3.30. L is solvable iff there exist a sequence of ideals $L = L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots \supseteq L_n = \{0\}$ such that $[L_i, L_i] \supseteq L_{i+1}$.

Exercise 3.31. Prove the above proposition.

We now state the main theorem of the section; compare the below to Engel's theorem:

Theorem 3.32 (Lie)**.** *Let L be solvable Lie algebra over an algebraically closed field of characteristic zero. Then, for any finite-dimensional representation V of L, there exists*

$$F : V_0 \subseteq V_1 \subseteq \cdots V_n = V$$

such that $\rho(L) \subseteq \mathfrak{b}(F)$.

Remark. The theorem is not true if $\operatorname{char} k > 0$. If $L = \mathfrak{sl}_2(\mathbb{Z}_2)$ is nilpotent, the standard representation does not have eigenvectors.

As with Engel's theorem, we state a variant:

Theorem 3.33 (Lie'). *Under the same assumptions, there exists a nonzero $v \in V$ such that $\rho(x)v = \chi(x)v$, where $\chi : L \to k$ is a linear functional.*

Exercise 3.34. Prove that Lie $\Leftrightarrow$ Lie'.

We now proceed to the proof of Lie's theorem.

PROOF OF LIE'S THEOREM. We shall need the following lemma, whose proof we postpone:

Lemma 3.35. *Let L be a Lie algebra over a field k of characteristic 0, I an ideal in L, V an L-module, v a nonzero element of V, and $\chi : L \to k$ a linear functional such that $yv = \chi(y)v$ for all y in$I$. Then $\chi([x, y]) = 0$ for all $y \in I$ and $x \in L$.*

Let $x \neq 0$. Let $V_i = \operatorname{span}\{v, xv, x^2v, \dots, x^{i-1}v\}$. Then $xV_i \subseteq V_{i+1}$. Let n be minimal integer such that $xV_n = V_{n+1}$. Then $\dim V_n = n$.

We shall make use of the following classical result:

Theorem 3.36 (Cayley-Hamilton). *Let v be a generic vector in an n-dimensional vector space V. If $A : V \to V$ is a linear transform, then*

$$A^n v + \lambda_1 A^{n-1} v + \cdots + 1 \cdot v = 0.$$

We now proceed by induction. $i = 0$ is okay. Assume for all integers $< n$

$$yx^i v = yxx^{i-1}v = xyx^{i-1}v - [x.y]x^{i-1}v.$$

By induction,

$$yx^{i-1}v = \chi(y)x^{i-1}v + v^1 \in V_{i-1}.$$

Note that

$$[x, y] \cdot x^{i-1}v = \chi([x, y])x^{i-1}v + v^n,$$

which is in V_{i-1}.

Now,

$$xyx^{i-1}v \in \chi(y)x^i v + xV_{i-1},$$

which is in V_{i-1}. This proves the claim.

We now prove the lemma. By the claim, all $y \in \operatorname{End} V_n$ is represented by a triangular matrix

$$\begin{pmatrix} \chi(y) & & * \\ & \ddots & \\ 0 & & \chi(y), \end{pmatrix}$$

with the basis $v, x_v, \dots, x^{n-1}v$. Therefore,

$$\operatorname{Tr}_{V_n}(y) = n\chi(n) = \operatorname{Tr}_{V_n}[x, y] = 0,$$

so $\chi[x, y] = 0$. This proves the lemma. □

Corollary 3.37. *If L is solvable, then*

$$L = L^{(1)} \supseteq L^{(2)} \supseteq \cdots \supseteq L^{(n)} = \{0\}$$

of Lie ideals, where $L^{(i)}/L^{(i+1)}$ is abelian.

Proof. $V = L$, $\rho = \mathrm{ad}$. □

Corollary 3.38. *If* $\operatorname{char} k = 0$ *and* L *is solvable, then* $[L, L]$ *is nilpotent.*

Exercise 3.39. Example : $\mathfrak{b} \subseteq \mathfrak{gl}_n$, $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$.

Exercise 3.40. If $[L, L]$ is nilpotent, then L is solvable.

Proof of corollary 3.38. If $k' \supseteq k$ is an extension, then $L' = L \otimes_k k'$ is a Lie algebra over k'. L is solvable iff L' is solvable (why?) So we may assume $k = \bar{k}$.

By corollary 3.37, there exists a filtration

$$L = L^{(1)} \supseteq L^{(2)} \supseteq \cdots \supseteq L^{(n)} = \{0\}$$

of ideals. Let $x \in [L, L]$. Then $\operatorname{ad} x(L^{(i)}) \subseteq L^{(i+1)}$. So $\operatorname{ad} x$ is nilpotent on $[L, L]$. Use theorem: if, for any x of a Lie algebra, $\operatorname{ad} x$ is nilpotent, then the algebra is nilpotent. □

5. Group Analog

Given a group G, we obtain the derived series

$$G = G^{(1)} \supseteq G^{(2)} \supseteq \cdots \supseteq G^{(k)} \supseteq \cdots,$$

where $G^{(2)} = (G, G) = \{xyx^{-1}y^{-1} : x, y \in G\}$ and $G^{(k+1)} = (G^{(k)}, G^{(k)})$. This is a normal series, and each $G^{(k)}/G^{(k+1)}$ is abelian.

Definition. A group G is *solvable* if $G^{(n)} = \{e\}$ for some n.

Example. $B \subseteq GL_n$, where B is the group of upper-triangular matrices.

Theorem 3.41. *Let* G *be a solvable, connected topological group. Let* $\rho : G \to GL(V)$*, where* V *is a finite-dimensional vector space over* $\mathbb{C}$*. Then there exists a flag*

$$\mathscr{F} : V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

such that $\rho(x)V_i \subseteq V_i$ *for all* i. □

Recall that V is irreducible over G if there are no proper invariant subspaces in V.

Corollary 3.42. *If* V *is irreducible, then* $\dim V = 1$. □

Corollary 3.43. *Any compact solvable group is abelian.* □

Theorem 3.44 (E. Cartan)**.** *Let* V *be a vector space over* k, $\operatorname{char} k = 0$, $\dim V < \infty$*. Lie algebra* $L \subseteq \mathfrak{gl}(V)$*. Then* L *is solvable iff* $\operatorname{Tr}_V(xy) = 0$ *for all* $x \in L$ *and* $y \in [L, L]$.

Exercise 3.45. Prove the above theroem. (STARRED! difficult!)

CHAPTER 4

Semisimple Lie Algebras

1. Some Linear Algebra

Let V be a vector space over k, $\bar{k} = k$, and $\operatorname{char} k = 0$, and $\dim V < \infty$.

Definition. $u \in \operatorname{End} V$ is *semisimple* if V has a basis $\{e_1, \ldots, e_n\}$ such that e_i is eigenvector for u,$i = 1, \ldots, n$.

Lemma 4.1. *For all $u \in \operatorname{End} V$, there exists a semisimple $s \in \operatorname{End} V$ and nilpotent $n \in \operatorname{End} V$ such that $u = s + n$ and $sn = ns$. Such presentation of u is unique.*

Proof. Let $P(t)$ be the characteristic polynomial of u. We can write

$$P(t) = \prod_i (t - \lambda_i)^{m_i}.$$

Set $V_i = \ker(u - \lambda_i)^{m_i}$. Then $V = \bigoplus V_i$, $\dim V_i = m_i$, and $uV_i \subseteq V_i$.

We now prove the uniqueness. Let $u = s + n$, $sn = ns$ implies that $su = us$, and so the eigenvalues of u and s are the same. But $u|_{V_i}$, the only eigenvalue is λ_i. s has the same eigenvalue so $s|_{V_i}$ is multiplication by λ_i. Therefore s is unique, and so n is unique.

How do we define s? We set $s|_{V_i}$ to be multiplication by λ_i. Then s is semisimple and $su = us$. And $n = u - s$ is nilpotent. □

2. Semisimple Lie Algebra

Let us first define the radical of a Lie algebra.

Exercise 4.2. Let L be a Lie algebra and $\mathfrak{a}$ and $\mathfrak{b}$ be solvable Lie ideals in L. Then $\mathfrak{a} + \mathfrak{b}$ is a solvable Lie ideal.

Corollary 4.3. *Let $\dim L < \infty$.Then L has a maximal solvable Lie ideal $\operatorname{rad} L$ which contains any solvable ideal.*

Definition. $\operatorname{rad} L$ is called the *radical* of L.

Definition. L is *semisimple* if $\operatorname{rad} L = 0$.

Exercise 4.4. L is semisimple if and only if L does not contain any nontrivial abelian ideals.

Example. $\mathfrak{gl}_n(k)$ is not semisimple.

Proof. Assume that L is not semisimple. Then $\operatorname{rad} L \neq 0$ and the derived series of $\operatorname{rad} L$ contains an abelian ideal. If L contains an abelian ideal $I \neq 0$, then $I \subseteq \operatorname{rad} L$. □

Theorem 4.5. *Lie algebra L is semisimple iff the Killing form $K(x, y) = \operatorname{Tr}(\operatorname{ad} x \cdot \operatorname{ad} y)$ is nondegenerate.*

Proof. Set $\mathfrak{n} = \{x \in L : K(x,y) = 0 \text{ for all } y\}$, the kernel of $K(x,y)$.

Exercise 4.6. $\mathfrak{n}$ is a Lie ideal.

$\mathfrak{n}$ is solvable (by Cartan's theorem), and so $\mathfrak{n}$ is in $\operatorname{rad} L$. If L is semisimple, then $\mathfrak{n} = \{0\}$.

Assume now that K is nondegenerate. Assume for a contradiction that L is not semisimple. Let $\mathfrak{a} \neq \{0\}$ be an abelian ideal in L. We set $\sigma = \operatorname{ad} x \circ \operatorname{ad} y \in \operatorname{End} L$. Fix $x \in \mathfrak{a}$ and $y \in L$. Note that $\sigma L \subseteq \mathfrak{a}$ and $\sigma\mathfrak{a} = 0$, and so $\sigma^2 = 0$. This implies that $\operatorname{Tr} \sigma = 0$, and so $\sigma \in \mathfrak{n}$, but since $\mathfrak{n} = \{0\}$, we have $\sigma = 0$. ☐

We now assume that $\dim L < \infty$.

Theorem 4.7. *Let L be a semisimple Lie algebra, and $\mathfrak{a}$ be a Lie ideal in L. Define*

$$\mathfrak{a}^{\perp} = \{x \in L : K(x,y) = 0 \text{ for all } y \in \mathfrak{a}\}.$$

Then $\mathfrak{a}^{\perp}$ is an ideal, and $L = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$.

Exercise 4.8. Prove the above theorem. Hint: $\mathfrak{a} \cap \mathfrak{A}^{\perp}$ is solvable.

Definition. L is *simple* if $L \neq \{0\}$ and L does not contain any proper ideals.

Theorem 4.9. *Any semisimple Lie algebra is a direct sum of simple Lie algebras.*

Proof. (1) Any simple Lie algebra is semisimple. (2) Use induction and the previous theorem. ☐

Exercise 4.10. If L is semisimple, then $L = [L, L]$.

Corollary 4.11. *If $L = \bigoplus L_\alpha$, L_α are simple Lie algebras, then any ideal in L is a direct sum of some L_α.*

Example. $\mathfrak{sl}(V)$ is simple if $\dim V \geq 2$.

Example. Assume $\dim V = 2n \geq 2$. Given nondegenerate skew-symmetric bilinear form $\Phi : V \times V \to k$, we define $sp(V)$ to be the Lie algebra of endomorphisms such that Φ is invariant.

Example. Given nondegenerate symmetric bilinear form $\psi : V \times V \to k$, let $\sigma(V)$ be the Lie algebra of endomorphisms such that ψ is invariant.

Note that $sl(V)$, $sp(V)$ are simple. $\sigma(V)$ is semisimple for $\dim V = 3$. $\sigma(V)$ is simple if $\dim V > 4$, and semisimple (depending on ψ) if $\dim V = 4$.

Henceforth, we let $\rho : L \to \mathfrak{gl}(V)$ be a representation.

Definition. V (or ρ) is *irreducible*, or *simple*, if $V \neq \{0\}$ and V does not have any proper L-submodules.

Definition. V is *semisimple*, or *completely reducible*, if V is isomorphic to a direct sum of simple modules.

Theorem 4.12 (H. Weyl). *If L is semisimple, then any finite-dimensional module over L is semisimple.*

The original proof of the above theorem by Hermann Wely, circa 1925, used analysis—a method known as the "unitary trick". We shall, however, present an algebraic proof, which was discovered in the 1950s.

We shall need two facts. First, we need "homological algebra". We say that a short sequence

$$0 \to A \xrightarrow{\varepsilon} B \xrightarrow{\pi} C \to 0$$

is *exact* if π is onto, ε is injective, and $\ker \pi = \operatorname{im} \varepsilon$. We say that the sequence *splits* if there exists σ such that $\pi\sigma = \mathrm{id}$ or there exists φ such that $\varphi\varepsilon = \mathrm{id}$. If the sequence splits, then $B \cong A \oplus C$, where the isomorphism is given by $(\varphi \oplus \pi)(b) = \varphi(b) \oplus \pi(b)$ or by $a \oplus c \mapsto \varepsilon(a) + \sigma(c)$.

Exercise 4.13. Check that these are isomorphisms and that φ exists if and only if σ exists.

We shall also need the construction of central elements of order two in $U(L)$. Such elements are called *Casimir elements*, or *Casimir operators*.

Proof. Let $B(x, y)$ be a nondegenerate, L-invariant[1], symmetric bilinear form on L. Let $\bar{e} = (e_1, \ldots, e_n)$ $\bar{f} = (f_1, \ldots, f_n)$ are biorthogonal bases in L, i.e.,

$$B(e_i, f_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We construct

$$b = \sum_{i=1}^{n} e_i f_i \in U(L).$$

Proposition 4.14. *b is central and b does not depend on $\bar{e}$ and $\bar{f}$.*

Proof of proposition. Construct $\Phi : L \otimes L \to \mathrm{Hom}(L, L)$ by setting

$$\Phi(x \otimes y)(z) = B(y, z)x$$

$L \otimes L$ and $\mathrm{Hom}(L, L)$ are L-modules

Exercise 4.15. Φ is an L-isomorphism.

Exercise 4.16. $\Phi(\sum e_i \otimes f_i) = \mathrm{id}$.

Then

$$\mathrm{Hom}_k(L, L) \xrightarrow{\Phi^{-1}} L \otimes_k L \xrightarrow{\text{product}} U(L),$$

and the image of id is b. THerefore, b does not depend on $\bar{e}$ or $\bar{f}$. Moreover,

$$[\mathrm{id}, \operatorname{ad} x] = 0$$

in $\mathrm{Hom}_k(L, L)$ for all $x \in L$, and so $[x, b] = 0$ in $U(L)$. It follows that b is in the center. □

Exercise 4.17 (Important!)**.** Prove that there is, up to scalar multiplication, a unique Casimir element for $\mathfrak{sl}_2(k)$. Here $\operatorname{char} k = 0$.

Lemma 4.18. *Let V be a finite-dimensional vector space V. If L is semismiple and $\rho : L \to \mathfrak{gl}(V)$ is injective, then $B_\rho(x, y) = \mathrm{Tr}(\rho(x)\rho(y))$ is nondegenerate.*

Proof of lemma. Let $\mathfrak{n} = \{x \in L : B_\rho(x, y) = 0 \text{ for all } y \in L\}$. Then $\mathfrak{n}$ is a solvable Lie subalgebra (Cartan's theorem), whence $\mathfrak{n} = \{0\}$ for L is semisimple. □

Lemma 4.19. *Let b be the Cashmir element defined by B_ρ. Then $U(L)$ acts on V and so $b : V \to V$. Then $\mathrm{Tr}_V\, b = \dim L$.*

---

[1] B is L-invariant if $B([z, x], y) + B(x, [z, y]) = 0$ for all $z \in L$.

We know that b is an L-homomorphism because b is central. Also,

$$\operatorname{Tr}(b) = \sum_i \operatorname{Tr}(\rho(e_i)\rho(f_i)) = \sum_i B_\rho(e_i, f_i) = \dim L$$

Lemma 4.20. *If V is simple, then b is an isomorphism.*

Proof of lemma. By Schur's lemma, b is an isomorphism or b is trivial. If b is trivial, then $\operatorname{Tr} b = 0$, a contradiction. $\square$

Lemma 4.21. *Let $0 \to V \xrightarrow{\varepsilon} W \xrightarrow{\pi} k \to 0$ be an exact sequence of L-modules (k-trivial L-module). Then the sequence splits.*

Proof of lemma. It is enough to consider the case when V is simple. In fact, the sequence splits when $\dim V = 1$. We induct on $\dim V$. If V is not simple, then $\{0\} \neq U \subsetneq V$, where U is an L-module. Then the sequence

$$0 \to V/U \to W/U \to k \to 0$$

is exact, and $W/U \cong V/U \oplus V'/U$, where $V' \cong u \oplus k$. This sequence splits, whence the sequence in question splits as well.

It is enough to prove when $\rho : L \to \mathfrak{gl}(V)$ is faithful. Assume ρ is not faithful, and let $\mathfrak{a} = \ker \rho$. Let $x \in \mathfrak{a}$. Then $xW \subseteq V$, for x acts trivially on k. Also, $xV = \{0\}$, for $x \in \ker \rho$. Therefore, $[\mathfrak{a}, \mathfrak{a}]$ acts trivially on W. But L is semisimple, and so $\mathfrak{a}$ is semisimple. It follows that $\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}]$. So the sequence in question splits over $L/\mathfrak{a}$ and V is a faithful representation of $L/\mathfrak{a}$.

We now proceed to prove the lemma. We may assume that V is simple, and that $\rho : L \to \mathfrak{gl}(V)$ is faithful. By Lemma 4.18, $B_\rho(x, y)$ is nondegenerate. Let $b : W \to W$ be the corresponding Casimir element. Now, $bW \subseteq V$, for b acts trivially on k. Furthermore, $b(V) = V$, for V is simple. Then $W \cong V \oplus \ker(b : W \to W)$, where $\dim \ker(b : W \to W) = 1$. It follows that the sequence in question splits. $\square$

We resume the proof of the theorem. It is enough to prove that any exact sequence of L-modules

$$0 \to E_1 \to E \to E_2 \to 0$$

splits. Let

$$W = \{\varphi \in \operatorname{Hom}_k(E, E) : \varphi|_{E_1} \text{ acts like scalar multiplication}\}$$

$$V = \{\varphi \in \operatorname{Hom}_k(E, E) : \varphi|_{E_1} = 0\}$$

Then we have sequence

$$0 \to V \xrightarrow{\varepsilon} W \xrightarrow{\pi} k \to 0$$

By Lemma 4.21, there exists $\varphi \in W$ such that φ is L-invariant and $\varphi|_{E_1} = \mathrm{id}$ ($W \cong V \oplus k$). $\square$

Exercise 4.22. Let L be Lie algebra, I a semisimple ideal in L. Then there exists an ideal $J \subseteq L$ such that $L = I \oplus J$.

Corollary 4.23. *If L is semisimple, then any derivation of L is $\operatorname{ad} x$ for some $x \in L$.*

Proof. Let $M = \operatorname{Der} L$. $M \supseteq L$, for $\operatorname{ad} x$ is a derivation. Note that L is an ideal in M, for $[D, \operatorname{ad} x] = -\operatorname{ad} Dx$. By Exercise 4.22, $M = L \oplus J$, J ideal, and $[L, J] = \{0\}$. We want $J = \{0\}$. We let $D \in J$, and observe that $\operatorname{ad}(Dx) = -[D, \operatorname{ad} x] = 0$, because $D \in J$. So Dx is in the center of L, but the center of L is trivial. $\square$

Corollary 4.24. *Let L be a finite-dimensional Lie algebra and I a semisimple Lie ideal. Then $L \cong I \oplus J$ for some Lie ideal J of L.*

Proof. We first observe that L is an I-module. By Weyl's theorem, we can find an I-invariant subspace J such that $L = I \oplus J$. Then $[I, J] \subseteq J$ because J is I-module, and $[I, J] \subseteq I$ because I is an ideal. It follows that

$$[I, J] \subseteq I \cap J = \{0\}.$$

We claim that $J = \{y \in L : [y, I] = 0\}$. In fact, for each $y \in J$, we can find $x \in I$ and $z \in J$ such that $y = x + Z$. Therefore,

$$[I, y] = [I, x] + [I, z] = [I, x] + 0 = [I, x].$$

It follows that $[y, J] = \{0\}$, whence $[x, J] = \{0\}$ as well. Therefore, x is in $Z(I)$, which is trivial because I is semisimple. We thus see that $y = z$, which is in J. We thus conclude that $L = I \oplus J$ as Lie algebras, which is the desired result. ☐

A consequence of the above corollary is that if L is Lie algebra and I a semisimple ideal, then

$$0 \to I \to L \to L/I \to 0$$

splits. Here is the dual statement:

Theorem 4.25 (Levy)**.** *Let $\pi : L_2 \to L_1$ be an epimorphism of Lie algebras and L_1 a semisimple Lie algebra. Then $L_2 \cong \ker \pi \oplus L_1$.*

Proof. Let $\mathfrak{a} = \ker \pi$. We wish to show that $L_2 \cong \mathfrak{a} \oplus L_1$.

Lemma 4.26. *If Levy's theorem is true when $\mathfrak{a}$ is abelian, simple, and nontrivial L_1-module, then the theorem is true for any $\mathfrak{a}$.*

Proof of lemma. We assume that $\mathfrak{a}$ contains a nontrivial ideal $\mathfrak{a}_1$. Then

$$0\mathfrak{a}/\mathfrak{a}_1 \to L/\mathfrak{a}_1 \to L_2 \to 0$$

$$0 \to \mathfrak{a}_1 \to \mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_1 \to 0$$

If Levy's theorem is true for these sequences, then $L_1 \cong \mathfrak{a} \oplus L_2$ (why?). In fact, $0 \to \mathfrak{a} \to L_2 \to L_1 \to 0$ splits.

So, we may assume that $\mathfrak{a}$ is a simple L_1-module. Note that $\pi(\operatorname{rad} L_2)$ is solvable in L_1. But L_1 is semisimple, and so $\pi(\operatorname{rad} L_2) = 0$. So $\operatorname{rad} L_2$ can either be $\mathfrak{a}$ or 0. If $\operatorname{rad} L_2 = \mathfrak{a}$, then $\mathfrak{a}$ is solvable. Take an ideal $[\mathfrak{a}, \mathfrak{a}]$ strictly contained in $\mathfrak{a}$, so $[\mathfrak{a}, \mathfrak{a}] = \{0\}$, so $\mathfrak{a}$ is abelian.

If L_2 acts trivially on $\mathfrak{a}$, then $\mathfrak{a}$ is in the center of L_2. We may define action of L_1 on $\mathfrak{a}$ as follows: for each $x \in L_1$, we define $\tilde{x} : \pi(\tilde{x}) = x$. We set the action of x on $\mathfrak{a}$ as $\tilde{x} \in L_2/$ IF $\pi(x') = x$, then $\tilde{x} - x' \in \mathfrak{a}$ acts trivially because $\mathfrak{a}$ is abelian.

We now conclude from Weyl's theorem that $L_2 \cong L_1 \oplus \mathfrak{a}$ (why?). ☐

Let $W = \operatorname{End} L_2$. W is an L_2-module, and the action is given by $\rho(x)f = \operatorname{ad} x \circ f - f \circ \operatorname{ad} x$. We construct L_2-submodules

$$\begin{aligned} P &= \{\operatorname{ad} x : x \in \mathfrak{a}\} \\ Q &= \{f \in W : f(L_2) \subseteq \mathfrak{a}, f(\mathfrak{a}) = \{0\}\} \\ R &= \{f \in W ; f(L_2) \subseteq \mathfrak{a}, f|_{\mathfrak{a}} = \text{ mult. by scalar}\}. \end{aligned}$$

Exercise 4.27. P, Q, and R are L_2-submodules.

We then have a short exact sequence

$$0 \to Q \to R \xrightarrow{\varphi} k \to 0.$$

We note that $\rho(x)R \subseteq P$. Then the induced sequence

$$0 \to Q/P \to R/P \xrightarrow{\tilde{\varphi}} k \to 0$$

is a sequence of L_1-modules ($L_2 = L_1/\mathfrak{a}$). Then there exists $w \in R/P$ such that $\tilde{\varphi}(w) = 1$ and w is L_1-invariant (why?) We set $I_w = \{x \in L_1 : xw = 0\}$. The theorem now follows from the following exercise (why?):

Exercise 4.28. $L_2 \cong \mathfrak{a} \oplus I_w$

□

Corollary 4.29. *Let L be a finite-dimensional Lie algebra. Then*

$$L \cong \operatorname{rad} L \oplus L_0,$$

where L_0 is semisimple.

Proof. $0 \to \operatorname{rad} L \to L \to L/\operatorname{rad} L \to 0$. Since $L/\operatorname{rad} L$ is semisimple, we have

$$L \cong \operatorname{rad} L \oplus L/\operatorname{rad} L$$

by Levy's theorem. □

Exercise 4.30. For $L = \mathfrak{gl}_n(k)$, find $\operatorname{rad} L$ and L_0.

CHAPTER 5

Representations of Special Linear Lie Algebras

1. Representations of $\mathfrak{sl}_2(\mathbb{C})$

We recall that

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the Lie bracket defined by

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

generates $\mathfrak{sl}_2(\mathbb{C})$.

Let V be an $\mathfrak{sl}_2$-module. We define

$$V^\lambda = \{v \in V : hv = \lambda v\},$$

the elements of *weight* λ.

Proposition 5.1. *If $v \in V^\lambda$, then $ev \in V^{\lambda+2}$ and $fv \in V^{\lambda-2}$.*

Proof. Observe first that

$$h(ev) = e(hv) + [h, e]v = e(\lambda v) + 2ev = (\lambda + 2)ev.$$

Similarly,

$$h(fv) = (\lambda - 2)fv,$$

which proves the proposition. □

Definition. An element $v \in V$ is *primitive of weight* λ if v is a nonzero element of V^λ such that $ev = 0$.

Proposition 5.2. *v is primitive if and only if it is $\mathfrak{b}_2$-invariant, where $\mathfrak{b}_2 = \operatorname{span}\{h, e\}$.*

Proof. If v is primitive, then v is $\mathfrak{b}_2$-invariant. Let $\mathbb{C}v$ be $\mathfrak{b}_2$-invariant, so that $ev = \mu v$ and $hv = \lambda v$. Then

$$[h, e]v = 2ev = 2\mu v$$

and

$$[h, e]v = (he - eh)v = 0,$$

which proves the proposition. □

Proposition 5.3. *Any finite-dimensional $\mathfrak{sl}_2$-module contains a primitive element.*

PROOF. Using Lie's theorem, we can furnish a one-dimensional $\mathfrak{b}$-invariant subspace, or say that for any eigenvector v for h the nonzero vectors of

$$v, ev, e^2v, \ldots$$

are linearly independent, for they have different eigenvalues. It now suffices to observe that the last nonzero vector $e^m v$ is primitive. □

Theorem 5.4. *Let V be a $\mathfrak{sl}_2$-module (not necessarily finite-dimensional) and v a primitive element of weight λ. Set*

$$v_n = \frac{1}{n!} f^n v$$

for each $n \geq 0$, and define $v_{-1} = 0$. Then

(i) $hv_n = (\lambda - 2n)v_n$
(ii) $fv_n = (n+1)v_{n+1}$
(iii) $evn = (\lambda - n + 1)v_{n-1}$.

PROOF. (i) follows from Proposition 5.1, (ii) by construction, and (iii) by induction on n. We first see that $ev_0 = 0$. Observe that

$$\begin{aligned} nev_n &= efv_{n-1} \\ &= [e,f]v_{n-1} + fev_{n-1} \\ &= (\lambda - 2n + 2)v_{n-1} + (\lambda - n + 2)(n-1)v_{n-1} \\ &= [(\lambda - n + 2)n - n]v_{n-1} \\ &= n(\lambda - n + 1)v_{n-1}, \end{aligned}$$

whence $ev_n = (\lambda - n + 1)v_{n-1}$. □

Corollary 5.5. *Let v_n be defined as in 5.4. Then one of the following is true:*

(a) $v_0, v_1, v_2, \ldots$ *nonzero linearly independent, or*
(b) $v_0, v_1, \ldots, v_m, 0, 0, \ldots$, *where* $v_0, \ldots, v_m$ *are nonzero. In this case,* $\lambda = m$.

PROOF. (a) Suppose $v_0, v_1, v_2, \ldots \neq 0$. The weights are different, so they are linearly independent.

(b) Assume that there are zero elements. Take the minimal m such that $v_{m+1} = 0$. Then $v_0, \ldots, v_m \neq 0$ and $ev_{m+1} = (\lambda - m)v_m$ imply $\lambda = m$. □

Corollary 5.6. *Let $\dim V < \infty$ and v a primitive element. Then*

$$W = \operatorname{span}\{v = v_0, v_1, \ldots, v_m\}$$

is an irreducible $\mathfrak{sl}_2$-submodule.

PROOF. (i), (ii), and (iii) show that W is an $\mathfrak{sl}_2$-submodule. The eigenvalues of h are $m, m-2, m-4, \ldots, -m$.

We claim that W is irreducible. In fact, we let W' be irreducible submodule of $W' \neq \{0\}$. Then, W' is h-invariant, so h has nonzero eigenvector $v_i \in W'$. Use powers of e: $v_i, v_{i-1}, \ldots, v_0 = v$. So $v \in W'$ implies $W' = W$. This proves the claim. □

We now fix a notation for $n \geq 0$:

$$\begin{aligned} W_m &= \operatorname{span}\{v_0, \ldots, v_m\} \\ hv_n &= (m-2n)v_n \\ fv_n &= (n+1)v_n \\ ev_n &= (m-n+1)v_{n-1} \end{aligned}$$

(We assume that $v_{m+1} = v_{-1} = 0$.) So, W_m is $\mathfrak{sl}_2$-module.

Theorem 5.7. *W_m is irreducible and any irreducible $\mathfrak{sl}_2$-module of dimension $m+1$ is isomorphic to W_m.*

Proof. By corollary 5.6, W_m is irreducible. Also, $\dim W_m = m+1$.

Let V be an irreducible $\mathfrak{sl}_2$-module with $\dim V = m+1$. We have already proved that V contains a primitive element v. Corollary 5.5 shows that $hv = m'v$, $m' \in \mathbb{Z}_{\geq 0}$ and that v generates irreducible submodule W of V such that $\dim W = m'+1$. Since V is irreducible, $V = W$ and $m' = m$. So, $V = W_m$. $\square$

Example. $W_0 = \mathbb{C}$, $W_1 = \mathbb{C}^2$ with standard representation, and $W_m = S^m(\mathbb{C}^2)$.

Corollary 5.8. *Any finite-dimensional $\mathfrak{sl}_2$-module is a direct sum of W_m*

Proof. This is a direct consequence of Weyl's theorem. $\square$

Example. Let M be a compact complex manifold of complex dimension n. Consider

$$V = \bigoplus_{i=0}^{n} H^i(M, \mathbb{C}).$$

If M is a Kähler manifold, then V is an $\mathfrak{sl}_2(\mathbb{C})$-module. In particular, if we take $hx = (n-p)x$, then $x \in H^p(M, \mathbb{C})$. This is a simple example of the so-called *Hodge theory*, which is based on the action of $\mathfrak{sl}_2(\mathbb{C})$.

2. Representations of $\mathfrak{sl}_n(\mathbb{C})$

Recall that $\mathfrak{sl}_n(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ containing matrices with trace zero. We have

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{h}$ is the diagonal matrices of trace zero. $\mathfrak{h}$ is called a *Cartan subalgebra.* $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is called a *Borel subalgebra.*

We now proceed to describe $\mathfrak{h}^*$, the collection of functionals $f : \mathfrak{h} \to \mathbb{C}$. Indeed, a typical element of $\mathfrak{h}^*$ is of the form

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \sum_{i=1}^{n} u_i \lambda_i,$$

where $u_1, \ldots, u_n \in \mathbb{C}$. This is defined up to adding $c \in \mathbb{C}$, for

$$\sum_{i=1}^{n} (u_i + c)\lambda_i = \sum_{i=1}^{n} u_i \lambda_i.$$

Then

$$\mathfrak{h}^* = \mathbb{C}^n / \mathbb{C} \cdot (1, \ldots, 1) \cong \mathbb{C}^{n-1}.$$

Definition. We define *positive roots*

$$R_+ \subseteq \mathfrak{f}^* : \{\Lambda \mapsto \lambda_i - \lambda_j, i > j\}$$

and *negative roots*

$$-(R_+) \subseteq \mathfrak{f}^* : \{\Lambda \mapsto \lambda_j - \lambda_i, i > j\}.$$

For any positive root, we define *simple roots* $\{\alpha_k\}$ such that

$$\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} = \lambda_i - \lambda_j,$$

where $\alpha_k = \lambda_{k+1} - \lambda_k$.

Now, let $\alpha = \lambda_i - \lambda_j$, and define

$$e_\alpha = E_{ij},$$

the matrix with 1 as the ijth entry and zero everywhere else, and

$$f_\alpha = e_{-\alpha} = E_{ij}.$$

We also define

$$h_\alpha = E_{ii} - E_{jj}.$$

Exercise 5.9. The following holds:

(a) $\{e_\alpha\}_{\alpha \in R_+}$ is a basis in $\mathfrak{n}_+$, and $\{e_{-\alpha}\}_{\alpha \in R_+}$ is a basis in $\mathfrak{n}_-$.
(b) $h \in \mathfrak{h}$ and $\alpha \in R$ yield $[h, e_\alpha] = \alpha(h) e_\alpha$.
(c) $[e_\alpha, e_{-\alpha}] = h_\alpha$.

CHAPTER 6

Theory of Weights and Roots

1. Weights and primitive elements

Let V be a finite-dimensional $\mathfrak{sl}_n$-module. For each $\chi \in \mathfrak{h}^*$, we set

$$V_\chi = \{v \in V : hv = \chi(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Proposition 6.1. *If $\alpha \in R$ and $v \in V_\chi$, then $e_\alpha v \in V_{\chi+\alpha}$.*

Proof.

$$h(e_\alpha v) = [h, e_\alpha]v + e_\alpha hv = \alpha(h)e_\alpha v + \chi(h)e_\alpha v = (\chi + \alpha)e_\alpha v.$$

□

Proposition 6.2. *If V is finite-dimensional, then $V = \bigoplus V_\chi$.*

Proof. Note that

$$W = \sum_{\chi \in \mathfrak{h}^*} V_\chi.$$

This is a direct sum (nonzero elements with different eigenvalues are linearly independent) and $V_\chi \neq \{0\}$. Then W is e_α-invariant and $\mathfrak{h}$-invariant, whence W is $\mathfrak{sl}_n$-invariant and W is a submodule. By Weyl's theorem, we can write $V = W \oplus V'$.

If $V' \neq \{0\}$, then we have an abelian Lie algebra. If $\mathfrak{h}$ acts on V', then there exists a nonzero $v \in V'$. v is $\mathfrak{h}$-eigenvector. Then $v \in W$. This is a contradiction, and so $V = W$. □

Example. Suppose $V = \mathfrak{sl}_n(\mathbb{C})$ and has the action of ad. We find all nonzero V_χ. In this case, $V = v_0 \bigoplus_{\alpha \in R} V_\alpha$. In particular, $V_0 = \mathfrak{h}$ with dimension $n-1$ and $V_\alpha = \mathbb{C}\alpha$. We leave it as an exercise to the reader to finish this proof.

We now introduce primitive elements.

Exercise 6.3. Let V be finite-dimensional, and $v \in V$. The following statements are equivalent.

(1) v is an eigenvector for $\mathfrak{h}$ and $e_\alpha v = 0$ for some $\alpha \in R^+$.
(2) v is an eigenvector for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$.

Definition. In the above case, the nonzero v is *primitive.*

Proposition 6.4. *Let V be a finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module. Then V has a primitive element.*

Proof. Use Lie's theorem (cf. $\mathfrak{sl}_2(\mathbb{C})$). □

Let us now consider irreducible $\mathfrak{sl}_n(\mathbb{C})$-modules. For convenience, we let $L = \mathfrak{sl}_n(\mathbb{C})$.

Theorem 6.5. *Let V be an L-module, v a primitive element in V, weight χ. Set $V_1 = U(L)v$, a submodule generated by v. Then*

(a) V_1 is irreducible.

(b) All weights of V_1 are of the form

$$\chi - \sum_{i=1}^{n-1} m_i \alpha_i,$$

where $m_i \in \mathbb{Z}_{\geq 0}$ and α_i simple roots.

(c) If $w \in V$ and the weight of w is χ, then $w = \mu v$ for some $\mu \in \mathbb{C}$.

We do not prove the above theorem. Instead, we consider the following

Theorem 6.6. *(a) Any irreducible finite-dimensional L-module V contains unique (up to a scalar multiplication) primitive vector v. In this case, the weight of v is called the* highest weight *of V.*

(b) Irreducible finite-dimensional modules of the same highest weight are isomorphic.

Proof. (a) A primitive element v exists. Assume weight of v is λ. Is it unique? We assume that there is another primitive element v' whose weight is χ'. Then, by Theorem 6.5 (b), we have

$$\begin{aligned} \chi' &= \chi - \sum_{i=1}^{n-1} m_i' \alpha_i; \\ \chi &= \chi' - \sum_{i=1}^{n-1} m_i \alpha_i; \end{aligned}$$

for $m_i' \geq 0$ and $m_i \geq 0$, for v and v' are primitive. Adding the two yields

$$0 = \sum_{i=1}^{n-1} (m_i + m_i')\alpha,$$

whence $m_i + m_i' = 0$ for each i, and so $m_i = m_i' = 0$. It follows that $\chi = \chi'$. By Theorem 6.5 (c), we have $v' = \mu v$ for some $\mu \in \mathbb{C}$. This proves (a).

(b) Given irreducible finite-dimensional L-modules V and V' with primitive elements v and v', we assume that both of their highest weights is χ. We consider the L-module $V \oplus V'$, whose primitive element is $w = (v, v')$. Note that the weight of w is χ.

We let $W \subseteq V \oplus V'$, the L-module generated by w. We note that $W \neq V \oplus V'$. Since $W \subseteq V \oplus V'$, we can consider the restriction of the projection maps $W \xrightarrow{\pi} V$ and $W \xrightarrow{\pi'} V'$.

We claim that π and π' are surjections. Indeed, $\pi(w) = v$ and $\pi'(w) = v'$. Since W is irreducible, π and π' are isomorphisms by Schur's lemma. It follows that $V \cong V'$, whence $V \cong W \cong V'$. □

We remark that the classification of irreducible finite-dimensional L-modules is equivalent to the classification of highest weights.

Let $\chi \in \mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{sl}_n(\mathbb{C})$. Note that $\lambda \in \mathfrak{h}$, where $\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $\lambda_1 + \cdots + \lambda_n = 0$, and

$$\chi(\lambda) = u_1\lambda_1 + u_2\lambda_2 + \cdots + u_n\lambda_n,$$

where $u_1, \dots, u_n$ are defined up to $+C$. We note that $u_i - u_j$ is defined uniquely.

Theorem 6.7. *There exists an irreducible finite-dimensional module with highest weight* χ *if and only if* $u_i - u_j \in \mathbb{Z}_{\geq 0}$ *for* $i < j$.

Proof. If V is irreducible and χ its highest weight, then $u_i - u_j \in \mathbb{Z}_{\geq 0}$ for $i < j$.

Exercise 6.8. Let $v \in V$ is primitive with highest weight χ. Set

$$v_m^\alpha = \frac{1}{m!}(e_{-\alpha})^m v$$

for $m \geq 0$. Then

(a) $e_{-\alpha} v_m^\alpha = (m+1) v_{m+1}^\alpha$:
(b) $h v_m^\alpha = (\chi - m_\alpha)(h) v_m^\alpha$, where $h \in \mathfrak{h}$.
(c) $e_\alpha v_m^\alpha = (\chi(h_\alpha) - m + 1) v_{m-1}^\alpha$.

Hint: We proved this for $\mathfrak{sl}_2$. *The rest of the proof is similar.*

Exercise 6.9. If V is finite-dimensional, then there exists $m \in \mathbb{Z}_{\geq 0}$ such that $v_m^\alpha \neq 0$, $v_{m+1}^\alpha = 0$. In this case,

$$\chi(h_\alpha) = m.$$

Hint: This is a corollary of the previous exercise. Again, look for $\mathfrak{sl}_2$. *Now, let* $\alpha = \alpha_{ij}$, *where* $i < j$. *Then*

$$u_i - u_j = \chi(h\alpha_{ij}) = m \geq 0,$$

which proves the first part.

We now assume that $u_i - u_j \in \mathbb{Z}_{\geq 0}$ for $i < j$. We shall construct V with highest weight χ.

Proposition 6.10. *If* χ *and* χ' *are highest-weight modules for* V *and* V', *respectively, then* $\chi + \chi'$ *is the highest-weight module for an irreducible* $W \subseteq V \otimes V'$.

Proof of proposition. Note that $\chi + \chi'$ is weight for $v \otimes v'$ and $v \otimes v'$ is primitive. Take $W = U(L) v \otimes v'$. Then W is irreducible with highest weight χ. We can add highest weights. □

Lemma 6.11. *Let* $\pi_i(\lambda) = \lambda_1 + \cdots + \lambda_i$, *i.e.,* $\pi(\lambda) = \lambda_1$. *If* χ *satisfies* $u_i - u_j \in \mathbb{Z}_{\geq 0}$ *for* $i < j$, *then*

$$\chi = \sum_{i=1}^{n-1} m_i \pi_i.$$

Proof of lemma. Note that

$$\begin{aligned} u_1\lambda_1 + \cdots + u_n\lambda_n &\equiv (u_1 - u_n)\lambda_1 + (u_2 - u_n)\lambda_2 + \cdots + (u_{n-1} - u_n)\lambda_{n-1} \\ &= \sum_{i=0}^{n-1} k_i \lambda_i, \end{aligned}$$

where $k_i \geq k_{i+1}$ for $i = 1, 2, \dots, n-2$; here $k_i \in \mathbb{Z}_{\geq 0}$.

It suffices to observe that

$$\begin{aligned}\sum_{i=0}^{n-1} k_i\lambda_i &= (k_1-k_2)\lambda_1+(k_2-k_3)(\lambda_1+\lambda_2)+(k_3-k_4)(\lambda_1+\lambda_2+\lambda_3)\\ &\quad +\cdots+k_{n-1}(\lambda_1+\cdots+\lambda_{n-1})\\ &= \sum_{i=0}^{n-1} m_i\pi_i,\end{aligned}$$

where $m_i \in \mathbb{Z}_{\geq 0}$. □

It is enough to construct an irreducible finite-dimensional V with highest weight π_i for $i = 1, \ldots, n-1$.

Proposition 6.12. *Let $V = \mathbb{C}^n$ with V standard representation. The highest weight of V is π_1 and primitive vector is*

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}.$$

Proof of proposition. Observe that

$$\begin{pmatrix}0 & & & *\\ & 0 & & \\ & & \ddots & \\ 0 & & & 0\end{pmatrix}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$$

and

$$\begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \lambda_1.$$

□

Exercise 6.13. Take $V_i = \Lambda^i V$, where $i = 1, \ldots, n-1$. Then the primitive vectors are $e_1 \wedge e_2 \wedge \cdots \wedge e_i$. The highest weight is π_i.

The desired result now follows from the exercise. □

Remark. Direct construction of V for

$$\sum_{i=1}^{n-1} m_i\pi_i.$$

If $\mathfrak{sl}_n = \mathfrak{sl}_2$, then we have already constructed V for π_1: this is the standard $\mathbb{C}^2$ with $m\pi_1 : \mathbb{S}^m\mathbb{C}^2$. In general, you apply $\mathbb{S}^m$.

2. Cartan Subalgebras

We now general semisimple Lie algebras. We shall generalize Cartan subalgebras and roots.

Let L be Lie algebra, and $\mathfrak{a}$ a Lie subalgebra. Normalizer of $\mathfrak{a}$ is

$$n(\mathfrak{a}) = \{x \in L : \operatorname{ad} x(\mathfrak{a}) \subseteq \mathfrak{a}\}$$

whence $\mathfrak{a}$ is an ideal in $n(\mathfrak{a})$. If $\mathfrak{a}$ is ideal in M, then $M \subseteq n(\mathfrak{a})$.

Definition. A subalgebra $\mathfrak{h} \subseteq L$ is *Cartan* if

(i) $\mathfrak{h}$ is nilpotent;
(ii) $\mathfrak{h} = n(\mathfrak{h})$.

Example. Let $L = \mathfrak{sl}_n$. Then

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right\},$$

where $\lambda_1 + \ldots + \lambda_n = 0$, is a Cartan subalgebra.

How do we construct Cartan subalgebras? Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. For $x \in L$, we define the *characteristic polynomial* to be

$$P_x(t) = \det(t - \operatorname{ad} x) = \sum_{i=0}^{n} a_i(x),$$

where $n = \dim L$ and $a_n(x) = 1$. Fix basis $\{e_1, \ldots, e_n\}$ in L, and write

$$x = \sum_{i=1}^{n} x_i e_i.$$

$a_i(x)$ is a homogeneous polynomial of degree $n-i$ in $x_1, \ldots, x_n$. Note that $a_{n-1}(x) = \operatorname{tr} \operatorname{ad} x$.

Definition. The *rank* of L is the minimal l such that $a_l(x) \neq 0$ for all $l \leq n$.

Note that $a_0(x) = 0$ because $a_0(x)$ is the product of eigenvalues of $\operatorname{ad} x$, but $\operatorname{ad} x$ has zero eigenvalue. Therefore, $1 \leq \operatorname{rank} L \leq n$.

Exercise 6.14. $\operatorname{rank} L = n$ if and only if L is nilpotent.

Definition. $x \in L$ is *regular* if $a_{\operatorname{rank} L}(x) \neq 0$.

We set

$$L_x^0 = \{y : (\operatorname{ad} x)^N y = 0 \text{ for some } N\}.$$

Exercise 6.15. $\dim L_x^0 = \operatorname{rank} L$ for all regular $x \in L$.

We set $L_r = \{x : x \text{ is regular}\}$.

Proposition 6.16. *$L_r \subseteq L$ is open, connected, and dense.*

PROOF. Let $V = \{x : a_{\operatorname{rank} L}(x) = 0\}$. V is closed, so L_r is open. $\dim_{\mathbb{C}} V = n - 1$, and so $\dim_{\mathbb{R}} V = 2n - 2$. Also, $L_r = L \setminus V$ is connected and dense. ☐

Let $x \in L$ and $\lambda \in \mathbb{C}$. We set L_x^λ be the collection of $x \in L$ such that $(\operatorname{ad} x - \lambda)^N = 0$ for some $N \in \mathbb{Z}$.

Proposition 6.17. *Let $x \in L$. Then*

(a) $L = \bigoplus_{\lambda \in \mathbb{C}} L_x^\lambda$,
(b) $[L_x^\lambda, L_x^\mu] \subseteq L_x^{\lambda+\mu}$, *where λ and μ are in $\mathbb{C}$,*
(c) L_x^0 *is a Lie subalgebra.*

Proof. (a) is an exercise in linear algebra.

$(b) \Rightarrow (c)$. Prove (b). Then $L \setminus V$ is open and connected, for it is of real codimension $V = 2$), and it is dense.

Let $x \in L$, $\lambda \in \mathbb{C}$. We set

$$L_x^\lambda \{y \in L : (\operatorname{ad} x - \lambda)^N y = 0 \text{ for some} N\}.$$

Then the following holds:

Exercise 6.18.

$$(\operatorname{ad} x - \lambda - \mu)^n [y, z] = \sum_{p=0}^{n} \binom{n}{p} [(\operatorname{ad} x - \lambda) 6p(y), (\operatorname{ad} x - \mu)^{n-p}(z)].$$

Let $y \in L_x^\lambda$, $z \in L_x^\mu$, $n \ll 0$. Then the following holds:

Exercise 6.19.

$$[(\operatorname{ad} x - \lambda) 6p(y), (\operatorname{ad} x - \mu)^{n-p}(z)].$$

equals zero. Therefore, $[y.z] \in L_x^{\lambda+\mu}$.

The desired result now follows from the above exercises. □

Let us now return to Cartan subalgebras.

Theorem 6.20. *If $x \in L$ is regular, then L_x^0 is Cartan and* $\dim L_x^0 = \operatorname{rank} L$.

Proof. We shall need the following

Exercise 6.21. $\dim L_x^0 = \operatorname{rank} L$. *Hint: This is an exercise in linear algebra.*

We shall prove that L_x^0 is Cartan.

Firstly, we claim that $n(L_x^0) = L_x^0$. In fact, we let $z \in n(L_x^0)$. Then $\operatorname{ad} z(L_x^0) \subseteq L_x^0$, which implies that $[z, x] \in L_x^0$.

By definition of L_x^0, there exists p such that $(\operatorname{ad} x)^p [z, x] = 0$. Therefore, we have $(\operatorname{ad} x)^{p+1}(z) = 0$, so that $z \in L_x^0$. This shows that $n(L_x^0) = L_x^0$.

Secondly, to prove that L_x^0 is nilpotent, it is enough to show that, for all $y \in L_x^0$, the map $\operatorname{ad} y|_{L_x^0}$ is nilpotent—this is a consequence of Engel's theorem. We set

$$\operatorname{ad}^1 y = \operatorname{ad} y|_{L_x^0} : L_x^0 \to L_x^0.$$

Then $\operatorname{ad}^2 y : L/L_x^0 \to L/L_x^0$ is a map of vector spaces.

We set U be the collection of $y \in L_x^0$ such that $\operatorname{ad}^1 y$ is not nilpotent and V the collection of $y \in L_x^0$ such that $\operatorname{ad}^2 y$ is invertible.

We shall need facts from linear algebra:

Lemma 6.22. *Let X and Y be finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. The the set of bijective maps $X \to Y$ is open in* $\operatorname{Hom}(X, Y)$. □

Lemma 6.23. *The set of not nilpotent maps in* $\operatorname{End} X$ *is open.* □

So, U and V are open in L_x^0, and $x \in V$ (why?). It thus follows that $V \neq \varnothing$. We wish to show that $U = \varnothing$.

Now, V is defined by a polynomial equation (why?) Working over $\mathbb{C}$, we get $\overline{V} = L_x^0$. If $u \neq \varnothing$, then $\overline{U} = L_x^0$ for the same reason, whence $U \cap V \neq \varnothing$.

Take $y \in U \cap V$. Then $\operatorname{ad}^1 y$ has zero as an eigenvalue, and so $[y, y] = 0$. In particular, the multiplicity of this eigenvalue is less than $\dim L_x^0 = \operatorname{rank} L$. Also, zero is not eigenvalue for $\operatorname{ad}^2 y$, for $\operatorname{ad}^2 y$ is invertible. It follows that the multiplicity of zero for the whole $\operatorname{ad} y$ is smaller than $\operatorname{rank} L$. This is a contradiction (why?). Therefore, $U = \varnothing$, and so L_x^0 is nilpotent. ☐

Theorem 6.24. *All Cartan subalgebras in L appear as L_x^0 for a regular x (This is not true for L over $\mathbb{R}$).* ☐

Corollary 6.25. *All Cartan subalgebras have the same dimension, namely* $\operatorname{rank} L$. ☐

Example. If $L = \mathfrak{sl}_n$, then the collection of diagonal matrices with trace 0 is a Cartan subalgebra.

We now consider semisimple Lie algebras.

Theorem 6.26. *Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ a Cartan subalgebra.*

(a) $\mathfrak{h}$ is abelian.
(b) All nonzero elements of $\mathfrak{h}$ are semisimple.
(c) The centralizer

$$c(\mathfrak{h}) = \{x \in L : \operatorname{ad} x(\mathfrak{h}) = 0\}$$

equals $\mathfrak{h}$, whence $\mathfrak{h}$ is maximally abelian.
(d) The restriction $K|_{\mathfrak{h}}$ of the Killing form K onto $\mathfrak{h}$ is nondegenerate.

Exercise 6.27. Find a nonabelian Cartan subalgebra.

Proof. Start with (d). There exists a regular $x \in L$ such that $\mathfrak{h} = L_x^0$. Note that

$$L = L_x^0 \oplus \sum_{\lambda \neq 0} L_x^\lambda$$

We shall make use of the following

Exercise 6.28. $K(L_x^\lambda, L_x^\mu) = 0$ if and only if $\lambda + \mu \neq 0$.

Then

$$L = L_x^0 \oplus \sum_{\lambda \neq 0} \left(L_x^\lambda \oplus L_x^{-\lambda}\right).$$

Since L is semisimple, K is nondegenerate, and so $K|_{L_x^0}$ and $K|_{L_x^\lambda} \oplus L_x^{-\lambda}$ are nondegenerate. This establishes (d).

We now prove (a). $\mathfrak{h}$ is nilpotent, so

$$K(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$$

by Cartan's theorem. But $K|_{\mathfrak{h}}$ is nondegenerate, and so $[\mathfrak{h}, \mathfrak{h}] = 0$. This proves (a).

We now prove (c). $\mathfrak{h}$ is abelian, so

$$\mathfrak{h} \subseteq c(\mathfrak{h}) \subseteq n(\mathfrak{h}) = \mathfrak{h}.$$

It follows that $\mathfrak{h} = c(\mathfrak{h})$, as desired.

We now prove (b). Let $x \in \mathfrak{h}$ and write $x = s + n$, where s is semisimple and n is nilpotent. Let $y \in \mathfrak{h}$. Then $[y, x] = 0$ for all $x \in \mathfrak{h}$, whence $[y, n] = 0$ (why?). So $[y, s] = 0$, and so $s, n \in c(\mathfrak{h}) = \mathfrak{h}$.

Since $[y, n] = 0$ and n is nilpotent, we have $\mathrm{Tr}(\operatorname{ad} y \circ \operatorname{ad} n) = 0$. Therefore, $K(y, n) = 0$ for all $y \in \mathfrak{h}$. Since K is nondegenerate, we have $n = 0$, as was to be shown. This completes the proof. $\square$

Corollary 6.29. *Cartan subalgebras is maximal abelian subalgebra of semisimple Lie algebra.* $\square$

Example. We give another example of maximal abelian subalgebras. If $L = \mathfrak{sl}_n$, then the collection of strictly upper triangular matrices is a maximal abelian subalgebra that is not Cartan.

Corollary 6.30. *Any regular element in a semisimple Lie algebra is semisimple.*

PROOF. $x \in L_x^0$. $\square$

Remark. Let L be a semisimple Lie algebra over $\mathbb{C}$. Then Any maximally abelian subalgebra consisting of semisimple elements is Cartan. In the real case, Theorems 6.20 and 6.26 are true, but Theorem 6.24 is not true.

3. Root System

Let V be a finite-dimensional vector space over $\mathbb{R}$ and α a nonzero vector in V. α defines a linear map $s_\alpha : V \to V$ satisfying

(i) $s_\alpha(\alpha) = -\alpha$
(ii) $H = \{x \in V : s_\alpha(x) = x\}$ is a hyperplane, i.e., a subspace of codimension 1.

The map s_α, which is uniquely defined, is called the *reflection* by α. We note that $s_\alpha^2 = \mathrm{id}$ and that $V \cong H \oplus \mathbb{R}\alpha$.

We let $\alpha^* \in V^*$ such that $\alpha^*|_H = 0$ and $\alpha^*(\alpha) = 2$.

Proposition 6.31. $s_\alpha(x) = x - \langle \alpha^*, x\rangle\alpha$, *where* $\langle \alpha^*, x\rangle = \alpha^*(x)$.

PROOF. Note that $s_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$. We furthermore have

$$s_\alpha(h) = h - \langle \alpha^*, h\rangle\alpha = h,$$

which proves the proposition. $\square$

Exercise 6.32. Prove that $\operatorname{End} V \cong V \otimes V^*$. Prove furthermore that $s_\alpha = 1 - \alpha \otimes \alpha^*$.

Lemma 6.33. *Take a finite subset R of V such that* $\operatorname{span} R = V$. *There is at most one reflection s_α such that $s_\alpha(R) = R$.*

PROOF. Let s_α and s'_α be such maps, and set $u = s_\alpha \cdot s'_\alpha$. Then $u(R) = R$, $u(\alpha) = \alpha$, and u defines an isomorphism $V/\mathbb{R}\alpha \xrightarrow{\cong} V/\mathbb{R}\alpha$.

Since R is finite, there exists an integer n such that $u^n(x) = x$ for all $x \in R$ (why?). Furthermore, all eigenvalues of u equal 1 and $u^n = \mathrm{id}$ on V (why?). It then follows that $u = \mathrm{id}$, whence $s'_\alpha = s_\alpha^{-1} = s_\alpha$, as was to be shown. $\square$

Definition. A subset R of V is a *root system* if

(i) R is a finite spanning set of V not containing 0.
(ii) For all $\alpha \in R$, there exists a reflection s_α such that $s_\alpha(R) = R$.

(iii) For all $\alpha, \beta \in R$, we have $s_\alpha(\beta) = \beta + n\alpha$ for some $n \in \mathbb{Z}$.

An element of R is called a *root* of V. The *rank* of R is the dimension of V.

For each root $\alpha \in R$, there is the dual root $\alpha^* \in V^*$, whence

$$s_\alpha(\beta) = \beta - \langle \alpha^*, \beta \rangle \alpha.$$

Therefore, condition (iii) above is equivalent to

$$\langle \alpha^*, \beta \rangle \in \mathbb{Z}.$$

We also note that $\alpha \in R$ implies $-\alpha \in R$, for $s_\alpha(\alpha) = -\alpha$.

Definition. A root system R is *reduced* if the only pair of collinear roots $(\alpha, -\alpha)$.

What if R is not reduced? Let α and β be collinear, so that $\beta = \mu\alpha$ for some $\mu \neq 1$. We may assume without loss of generality that $\mu > 0$, for $\alpha \in R$ implies $-\alpha \in R$. Furthermore, we may also assume that $\mu < 1$, by swapping the roles of α and β if necessary. Condition (iii) now implies that

$$-\mu = s_\alpha(\mu\alpha) = \mu\alpha + n\alpha,$$

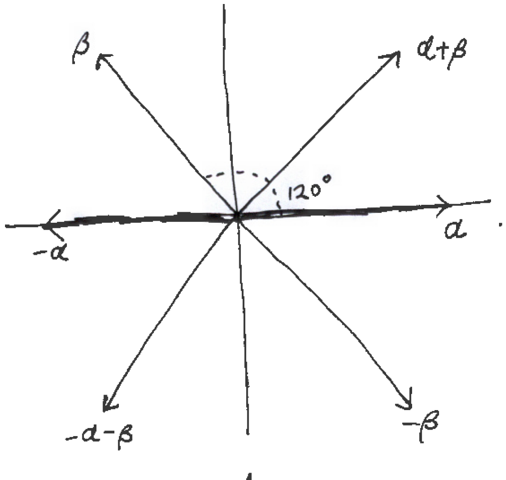
and so $2\mu \in \mathbb{Z}$. We therefore have $\mu = \frac{1}{2}$.

Remark. For semisimple Lie algebras over $\mathbb{C}$, we only need reduced root systems—not so for algebras over $\mathbb{R}$. Here we consider complex Lie algebras as vector space over $\mathbb{R}$.

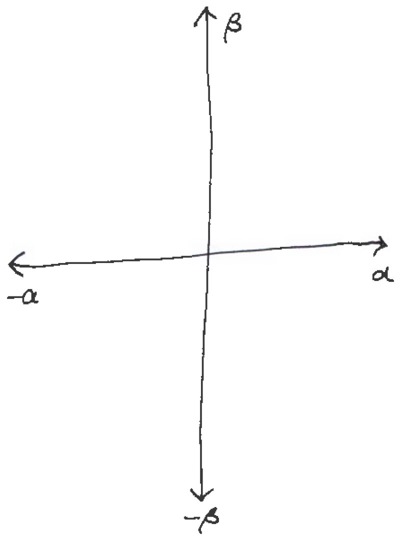
We now consider examples of reduced root systems.

Example. If $\operatorname{rank} R = 1$, then $\dim V = 1$, and so $R = \{\alpha, -\alpha\}$.

Example. If $\operatorname{rank} R = 2$, then:



A_2
($\mathfrak{sl}_3$)



$A_1 \cdot A_1$

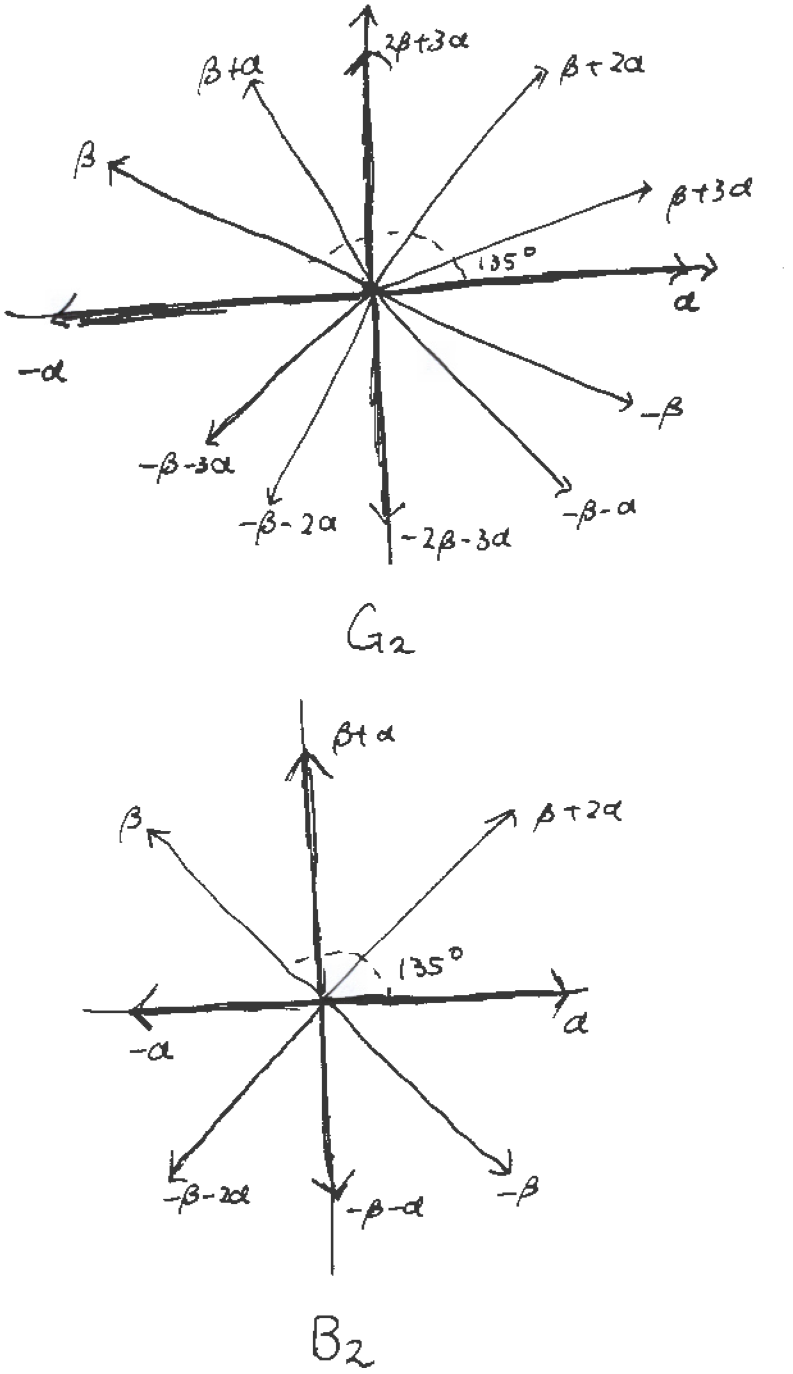


Figure 1. Reduced root systems of rank 2

Exercise 6.34. Make a nonreduced system out of B_2. Can we do this with A_2 or G_2?

4. Weyl Groups

Let V be a vector space over $\mathbb{R}$ and R a root system of R. The *Weyl group* $W \subseteq GL(V)$ is generated by reflections s_α for each $\alpha \in R$. We note that W can be identified with subgroups of $\operatorname{Aut} R$. We note that W is a *Coxeter group*, for the generators satisfy the relation $s_\alpha^2 = 1$.

Exercise 6.35. Find W for reduced systems of rank 2.

Let us now introduce an inner product on V.

Proposition 6.36. *There exists a nondegenerate, symmetric, positive-definite, W-invariant bilinear form on V.*

Proof. We take any nondegenerate, symmetric, positive-definite bilinear form $B : V \times V \to \mathbb{R}$ and set

$$(x, y) = \sum_{w \in W} B(wx, wy).$$

This defines the desired bilinear form (why?) ☐

The inner product $(\cdot, \cdot)$ defines an isomorphism $V \xrightarrow{\cong} V^*$. Note also that we have the dual root system

$$R^* = \{\alpha^* : \alpha \in R\}$$

for V^*.

Exercise 6.37. Prove that

$$s_\alpha(x) = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha$$

for all $x \in V$. This, in particular, shows that condition (iii) is equivalent to

$$2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

for all $\alpha, \beta \in R$.

As usual, the inner product $(\cdot, \cdot)$ defines angles:

$$(x, y) = |x||y| \cos\varphi.$$

We can then find angles between two roots α and β. Indeed,

$$n(\beta, \alpha) = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = 2\frac{|\alpha||\beta| \cos\varphi}{|\alpha|^2} = 2\frac{|\beta|}{|\alpha|}\cos\varphi,$$

and so $n(\beta, \alpha)n(\alpha, \beta) = 4\cos^2\varphi$ must be an integer. Therefore, the possible cases are

$$\cos^2\varphi = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}.$$

If $\cos^2\varphi = 1$, then α and β are collinear, which leads to the trivial case.

$n(\alpha, \beta)$	$n(\beta, \alpha)$	φ	$\lvert\beta\rvert/\lvert\alpha\rvert$
0	0	$\pi/2$	
1	1	$\pi/3$	1
1	-1	$2\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
-1	-2	$3\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-3	$5\pi/6$	$\sqrt{3}$

Proposition 6.38. *Let α and β be noncolliniear roots. If $n(\beta,\alpha) > 0$, so that $(\alpha,\beta) > 0$ and $\varphi < \pi/2$, then $\alpha - \beta$ is a root.*

Proof. If $n(\beta,\alpha) > 0$, then $n(\beta,\alpha) = 1$ or $n(\alpha,\beta) = 1$. If $n(\beta,\alpha) = 1$, then

$$\alpha - \beta = -(\beta - n(\beta,\alpha)\alpha) = -(\beta - \alpha^*(\beta)) = -s_\alpha(\beta),$$

which is in R. If $n(\alpha,\beta)$, then $\alpha - beta = s_\beta(\alpha)$, which is also in R. This proves the proposition. ☐

5. Simple Roots

Definition. $S \subseteq R$ is a *basis*, or a set of *simple roots*, if

(i) S is a basis for V.
(ii) For all $\beta \in R$, we have

$$\beta = \sum_{\alpha \in S} m_\alpha \alpha,$$

where $m_\alpha \in \mathbb{Z}$. Furthermore, all m_α must have the same sign.

Exercise 6.39. Find the simple roots of $\mathfrak{sl}_n(\mathbb{C})$.

Theorem 6.40. *Every root admits a set of simple roots.* ☐

We shall construct the simple roots as follows. Let $t \in V^*$ such that $\langle t, \beta\rangle \neq 0$ for all $\beta \in R$. We set

$$R_t^+ = \{\beta \in R : \langle t, \beta\rangle > 0\}.$$

Then $R = R_t^+ \cup (-R_t^+)$.

Definition. $\beta \in R$ is *reducible* if $\beta = \gamma_1 + \gamma_2$ for some $\gamma_1, \gamma_2 \in R_t^+$.[1] Otherwise β is *irreducible.*

Denote by $S_t \subseteq R_t^+$ the set of irreducible elements. We are now ready to prove the following more general

Theorem 6.41. *S_t is a basis. Whenever S is a basis, there exists $t \in V^*$ such that $S_t = S$.*

Proof. We prove first that S_t is a basis.

Lemma 6.42. *Each element of R_t^+ is a linear combination of elements from S_t with nonnegative coefficients.*

Proof of lemma. Let $I \subseteq R_t^+$ be the set of elements that cannot be written in this way, and suppose for a contradiction that I is nonempty. Then there exists $\alpha \in I$ such that $\langle t, \alpha\rangle$ is minimal. Note however that $\alpha \notin S_t$, for $\alpha = 1 \cdot \alpha$. Therefore, $\alpha = \beta + \gamma$ for some $\beta, \gamma \in R_t^+$, whence

$$\langle t, \alpha\rangle = \langle t, \beta\rangle + \langle t, \gamma\rangle.$$

Since $\langle t, \beta\rangle > 0$ and $\langle t, \gamma > 0$, we have $\langle t, \beta\rangle < \langle t, \alpha\rangle$. Therefore, $\beta \notin I$ and $\gamma \notin I$, whence β and γ can be written as a sum

$$\sum_{\gamma \in S_t} m_\gamma \gamma$$

for $m_\gamma \geq 0$. It implies that α can also be written in this way, which is evidently absurd. It thus follows that $I = \varnothing$, which proves the lemma. ☐

---

[1] We remark that R_t^+ is a cone.

Lemma 6.43. *$(\alpha, \beta) \leq 0$ if $\alpha, \beta \in S$, where $(\cdot, \cdot)$ is any W-invariant, symmetric, nondegenerate, positive-definite bilinear form.*

Proof of lemma. We have shown that, if $(\alpha, \beta) > 0$, then $\gamma = \alpha - \beta$ is a root. If $\gamma \in R_t^+$, then $\alpha = \beta + \gamma$ is reducible in R_t^+, so that $\alpha \notin S_t$. This is evidently absurd, and so $\gamma \in -R_t^+$.

It follows that $\beta = \alpha + (-\gamma)$ is reducible in R_t^+, which is also a contradiction. We now conclude that $(\alpha, \beta) \leq 0$. ☐

Lemma 6.44. *Let $t \in V^*$ and A a finite subset of V such that*

(a) $\langle t, \alpha \rangle > 0$ for all $\alpha \in A$.
(b) $(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in A$.

Then the elements of A are linearly independent[2].

Proof of lemma. Any relation in A can be written as

$$\sum y_\beta \beta = \sum z_\gamma \gamma,$$

where $y_\beta, z_\gamma > 0$. We set

$$\lambda = \sum y_\beta \beta,$$

so that

$$(\lambda, \lambda) = \sum y_p z_\gamma (\beta, \gamma).$$

Since $y_\beta z_\gamma > 0$ and $(\beta, \gamma) \leq 0$, we have $(\lambda, \lambda) \leq 0$. But $(\cdot, \cdot)$ is positive-definite, so that $(\lambda, \lambda) = 0$, whence $\lambda = 0$. It follows that the elements of A are linearly independent. ☐

Lemma 6.42 and Lemma 6.44 show that S_t is a basis.

We now proceed to prove the second part. To this end, we let S be a basis. There exists $t \in V^*$ such that $\langle t, \alpha \rangle > 0$ for all $\alpha \in S$. Set

$$R^+ = \left\{ \beta \in R : \beta = \sum_{\alpha \in S} m_\alpha \alpha, \alpha \in S, m_\alpha, m_\alpha \geq 0 \right\}.$$

Then $R^+ \subseteq R_t^+$ and $(-R^+) \subseteq -R_t^+$. We have $R^+ = R_t^+$ (why?).

Since S is a basis, all elements of S are irreducible, and so $S \subseteq S_t$. We now note that $|S| = |S_t| = \dim V$, whence $S = S_t$. This completes the proof. ☐

Example. We now apply the theorem to vector spaces of dimension 2. We assume that $\dim V = 2$ and find a basis $\{\alpha, \beta\}$ for R. Then the angle between α and β must be at least $\pi/2$. We have investigated all possible cases already: $A_1 \times A_1$, A_2, B_2, G_2.

Definition. Let R be a root system and S a basis. A *positive root* is a root β that can be written as the sum

$$\beta = \sum m_\alpha \alpha,$$

where $\alpha \in S$ and $m_\alpha \geq 0$. The set of positive roots is denoted by R^+.

---

[2] In our case, $A = St$.

Proposition 6.45. *Any positive root β can be written as a finite sum*

$$\beta = \alpha_1 + \cdots + \alpha_k$$

of simple roots α_i, and each partial sum

$$\alpha_1 + \cdots + \alpha_j$$

is a positive root.

Proof. Choose $t \in V^*$ such that $\langle t, \alpha \rangle = 1$ for all $\alpha \in S$ (why can we do this?). Let $\beta \in R^+$, so that $\langle t, \beta \rangle \in \mathbb{Z}_{>0}$. We proceed by induction on $m = \langle t, \beta \rangle$. If $m = 1$, then $\beta \in S$, and there is nothing to prove.

We now assume that $m > 1$. Then $\beta \notin S$. If β is collinear to an element $\alpha \in S$, then $\beta = 2\alpha = \alpha + \alpha$. Take $\gamma = \beta - \alpha$. We know that $(\alpha, \beta) > 0$ and that γ is a root. Therefore,$\gamma \notin (-R^+)$, for otherwise $\alpha = \beta + (-\gamma)$ but α is irreducible. So, $\gamma \in R^+$ and

$$\langle t, \gamma \rangle = \langle t, \beta \rangle - \langle t, \alpha \rangle = m - 1.$$

By induction, we proved the proposition for γ, whence the proposition is true for β as well. ☐

Proposition 6.46. *Let R be a reduced system of roots and S a basis. For any $\alpha \in S$, the image $S_\alpha(R^+ \smallsetminus \{\alpha\})$ is contained in $\subseteq R^+ \smallsetminus \{\alpha\}$.*

Proof. Let $\beta \in R^+ \smallsetminus \{\alpha\}$. Then

$$\beta = \sum_{\gamma \in S} m_\gamma \gamma$$

for $m_\gamma \geq 0$. Since R is reduced, β is not collinear to α, whence we can find $\gamma \in S$ such that $\gamma \neq \alpha$ and $m_\gamma > 0$.

Observe that

$$s_\alpha(\beta) = \beta - n(\beta, \alpha)\alpha$$

are the coefficients m_γ, $\gamma \neq \alpha$, for β and $s_\alpha(\beta)$ are equal. Then $s_\alpha(\beta) \neq \alpha$ and $s_\alpha(\beta) \in R^+ \smallsetminus \{\alpha\}$. ☐

Corollary 6.47. *Set*

$$\rho = \frac{1}{2} \sum_{\gamma \in R^+} \gamma.$$

Then $s_\alpha(\rho) = \rho - \alpha$.

Proof. Set

$$\rho_\alpha = \frac{1}{2} \sum_{R^+ \smallsetminus \{\alpha\}} \gamma.$$

Then $s_\alpha(\rho_\alpha) = \rho_\alpha$ (why?). Now, $\rho = \rho_\alpha + \frac{\alpha}{2}$, and so

$$s_\alpha(\rho) = s_\alpha(\rho_\alpha) + \frac{1}{2} s_\alpha(\alpha) = \rho_\alpha - \frac{\alpha}{2} = \rho - \frac{\alpha}{2} - \frac{\alpha}{2} = \rho - \alpha.$$

☐

Exercise 6.48. Check the corollary for $A_1 \times A_1$, B_2, and G_2.

Proof. If R is reduced, then S^* is a basis. ☐

Exercise 6.49. Find S^* when $\operatorname{rank} R = 2$.

6. Weyl Groups and Reduced Root Systems

Let W be a Weyl group, R a reduced root system, and S a basis.

Theorem 6.50. *The following holds:*

- *(i) For all $t \in V^*$, there exists $w \in W$ such that $\langle w(t), \alpha\rangle \geq 0$ for all $\alpha \in S$.*
- *(ii) For any basis S' of R, there exists a unique element $w \in W$ such that $w(S') = S$.*
- *(iii) For all $\beta \in R$, there exists $w \in W$ such that $w(\beta) \in S$.*
- *(iv) W is generated by $\{s_\alpha : \alpha \in S\}$.*

Proof. Let $W_S = \langle\{s_\alpha : \alpha \in S\}\rangle \subseteq W$. We shall prove (1), (2), and (3) for $w \in W_S$.

(1) Let $t \in V^*$. Recall that

$$\rho = \frac{1}{2}\sum_{\beta\in R^+}\beta,$$

and that $s_\alpha(\rho) = \rho - \alpha$ whenever $\alpha \in S$. Fix $w \in W_S$ that maximizes $\langle w(t), \rho\rangle$. then

$$\langle w(t), \rho\rangle \geq \langle s_\alpha w(t),\ rho\rangle = \langle w(t), s_\alpha\rho\rangle = \langle w(t), \rho - \alpha\rangle = \langle w(t), \rho\rangle - \langle w(t), \alpha\rangle.$$

Therefore, $\langle w(t), \alpha\rangle \geq 0$.

(2) We shall only show the existence (why do we have uniqueness?). Let $t' \in V^*$ such that $\langle t', \alpha'\rangle > 0$ for all $\alpha' \in S'$. (1) furnishes $w \in W_S$ such that $\langle w(t'), \alpha\rangle \geq 0$ for all $\alpha \in S$. Set $t = w(t')$, and observe that

$$0 \leq \langle w(t'), \alpha\rangle = \langle t', w^{-1}\alpha\rangle.$$

Since $\langle t', \alpha'\rangle > 0$ for all $\alpha' \in S'$, we have $\langle t', r\angle \neq 0$ for any root r. This implies that $\langle t', w^{-1}\alpha\rangle > 0$ and $\langle t, \alpha\rangle > 0$. Theorem 6.41 now implies that $S = S_t$ and $S' = S_{t'}$. Since $w(t') = t$, it follows that $w(S_{t'}) = S_t$.

(3) Let $\beta \in R$ and L_β the corresponding hyperplane, so that $s_\beta(L_\beta) = L_\beta$. Note that if $\gamma \neq \pm\beta$, then $L_\gamma \neq L_\beta$. There exists a finite set of hyperplanes, and so there exists $t_0 \in L_\beta$ such that $t_0 \notin L_\gamma$ for all $\gamma \neq \pm\beta$. This implies that $\langle t_0, \beta\rangle \neq 0$ and $\langle t_0, \gamma\rangle \neq 0$ for $\gamma \neq \pm\beta$. It follows that

$$|\langle t_0, \beta\rangle| > \langle t_0, \beta\rangle.$$

We can now find $t \in V^*$ close to t_0 such that

$$.|\langle t, \gamma\rangle| > \langle t, \beta\rangle > 0.$$

So, $\langle t, \gamma\rangle \neq 0$. Then $\beta \in S_t$ (why?). By (2), there exists $w \in W_S$ such that $w(S_t) = S$, whence $w(\beta) \in S$.

(4) We now show that $W_S = W$. To this end, it is enough to prove that $s_\beta \in W_S$ for all $\beta \in R$. By (3), there exists $w \in W_S$ such that $w(\beta) \in S$. Let $\alpha = w(\beta)$. Then $s_\alpha = s_w(\beta) = ws_\beta w^{-1}$, and so $w^{-1}s_\alpha w = s_\beta$. It follows that $s_\beta\ inW_S$. □

Definition. The *Weyl chamber* of W associated with S is

$$C_W = \{t \in V^* : \langle t, \alpha\rangle > 0, \alpha \in S\}.$$

By Theorem 6.50, C_W is a connected component of

$$V^* \smallsetminus \bigcup_{\beta \in R} L_\beta.$$

Exercise 6.51. Find all C_W for systems of rank 2.

We shall show that W is a Coxeter group. More specifically, we shall show that

$$W = \langle \{s_\alpha : \alpha \in S\} : s_\alpha^2 = (s_\alpha, s_\beta)^{m(\alpha,\beta)} = 1 \rangle,$$

where $(\cdot,\cdot)$ is the group commutator and

$m(\alpha,\beta)$	$\varphi(\alpha,\beta)$
2	$\pi/2$
3	$2\pi/3$
4	$3\pi/4$
6	$5\pi/6$

Definition. Recall that

- $s_\beta(\alpha) = \alpha - n(\alpha,\beta)\beta$
- $n(\alpha,\beta) = \langle \beta^*, \alpha \rangle \in \mathbb{Z}$
- $n(\alpha,\alpha) = 2$,
- $n(\alpha,\beta) = 0, -1, -2, -3$

if $\alpha, \beta \in S$. The *Cartan matrix* is

$$(n(\alpha,\beta))_{\alpha,\beta \in S}.$$

Example. For G_2, the Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Proposition 6.52. *A reduced root system is defined uniquely, up to an isomorphism, by its Cartan matrix.* □

Proposition 6.53. *Given two reduced systems (V, R, S) and (V', R', S'), we define a bijection $\varphi : S \to S'$ such that $n(\alpha,\beta) = n(\varphi(\alpha), \varphi(\beta))$ for all $\alpha, \beta \in S$. Then there exists a unique isomorphism $f : V \to V'$ such that $f(R) = R'$ and $f|_S = \varphi$.*

Proof. f is defined by linearity. f is an isomorphism, so if $\alpha, \beta \in s$, then

- $s_{\varphi(\alpha)}(f(\beta)) = s_{\varphi(\alpha)}$
- $(\varphi(\beta)) = \varphi(\beta) - n(\varphi(\beta), \varphi(\alpha))\varphi(\alpha) = \varphi(\beta) - n(\beta,\alpha)\varphi(\alpha)$.
- $f(s_\alpha(\beta)) = f(\beta - n(\beta,\alpha)\alpha) = \varphi(\beta) - n(\beta,\alpha)\varphi(\alpha)$.

It then follows that $s_{\varphi(\alpha)} \circ f = f \circ s_\alpha$, whence $W' = fWf^{-1}$ for the corresponding Weyl groups. We know that $R = W(S)$ and $R' = W'(S)$, whence $f(R) = R'$. □

Cartan matrices give rise to *Coxeter graphs*. Given (R, S), we construct a nondirected graph G as follows. We take S to be the vertices, and the number of edges between α and β is taken to be

$$n(\alpha,\beta) \cdot n(\beta,\alpha) = 0, 1, 2, 3.$$

If $(R, S) \cong (R', S')$, then the corresponding Coxeter graphs are isomorphic.

Example. Some examples of Coxeter graphs:

Definition. Let R be a root system in V. We say that R is the *sum* of R_1 and R_2 if $V = V_1 \oplus V_2$, R_i a root system in V_i, and $R = R_1 \cup R_2$.

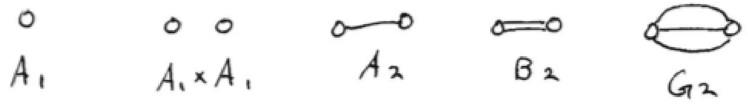


FIGURE 2. Coxeter graphs

Proposition 6.54. *In this case, $V_1 \perp V_2$.*

PROOF. Let $\alpha \in R_1$ and $\beta \in R_2$, so that $\alpha \cdot \beta \notin V_1 \cup V_2$. Then $\alpha - \beta$ is not a root, and so $(\alpha, \beta) \leq 0$. Similarly, $\alpha + \beta$ is not a root, and so $(\alpha, -\beta) \leq 0$, or $(\alpha, \beta) \geq 0$. Therefore, $\alpha \perp \beta$. Since $V_i = \operatorname{span}\{R_i\}$, we have $V_1 \perp V_2$. ☐

Definition. R is *irreducible* if R cannot be written as a union of nontrivial root systems.

Corollary 6.55. *Any root system is a sum of irreducible root systems.* ☐

Corollary 6.56. *R is irreducible if and only if its Coxeter graph is connected.*

PROOF. If $R = R_1 \cup R_2$, then we may choose $S = S_1 \cup S_2$: for example, we take S_i to be a basis of R_i. By Proposition 6.54, $S_1 \perp S_2$, whence there are no edges connecting S_1 and S_2.

Conversely, if $S = S_1 \cup S_2$ such that there are no edges between S_1 and S_2, then $S_1 \perp S_2$ and $V_1 \perp V_2$, where $V_i = \operatorname{span} S_i$. We also note that $s_\alpha(S_i) \subseteq S_i$ for all $\alpha \in S$, whence $R \subseteq V_1 \cup V_2$ (why?). It follows that R is reducible. ☐

Theorem 6.57. *Any such graph is isomorphic to one of the following:*

SKETCH OF PROOF. We define a symmetric bilinear form $A(x, y)$ on $\mathbb{R}^S$ as follows: for a fixed basis $(e_\alpha)_{\alpha \in s}$, we set $A(e_\alpha, e_\alpha) = 1$ and

$$A(e_\alpha, e_\beta) = \begin{cases} \cos(\pi/2) & \text{if there are 0 edges;} \\ \cos(2\pi/3) & \text{if there is 1 edge;} \\ \cos(3\pi/4) & \text{if there are 2 edges;} \\ \cos(5\pi/6) & \text{if there are 3 edges.} \end{cases}$$

It then suffices to show that G is Coxeter if and only if A is a nondegenerate, positive-definite form. ☐

We now define Dynkin diagrams, which are Coxeter graphs with extra data. The extra data is the vertex $\alpha \mapsto$ number proportional to (α, α).

Theorem 6.58. *Dynkin diagrams define Cartan matrix (up to an isomorphism)*

SKETCH OF PROOF. We construct the Cartan matrix as follows:

If	Then
$\alpha = \beta$	$n(\alpha, \beta) = 2$
$\alpha \neq \beta$, no edge	$n(\alpha, \beta) = 0$
$\alpha \neq \beta$, $\exists$ edge and $(\alpha, \alpha) \leq (\beta, \beta)$	$n(\alpha, \beta) = 1$
$\alpha \neq \beta$, $\exists$ i edges $(i = 1, 2, 3)$ and $(\alpha, \alpha) > (\beta, \beta)$	$n(\alpha, \beta) - i$

We invite the reader to fill in the details. ☐

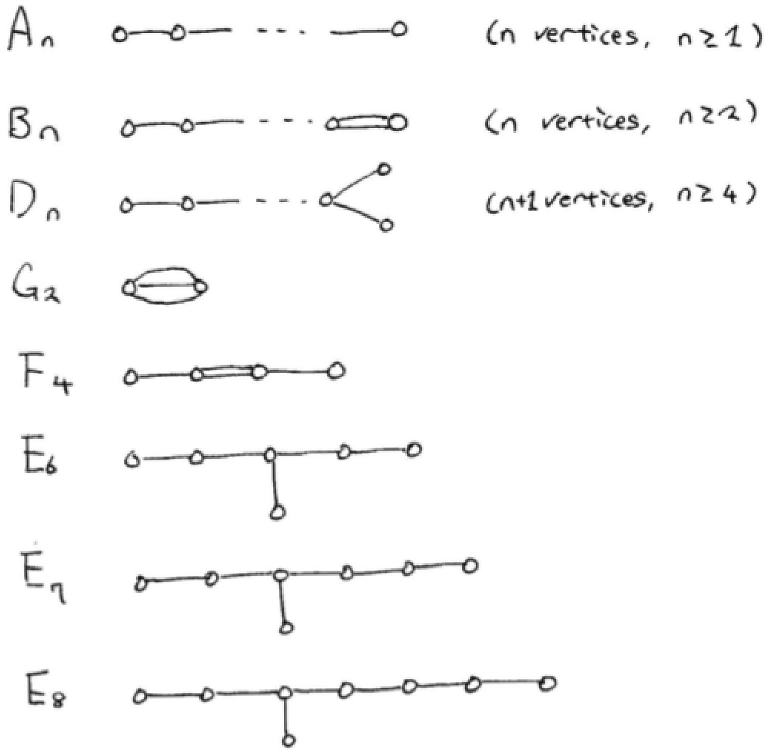


Figure 3. Classification of Coxeter graphs

7. Constructions of Irreducible Root Systems

Let $\{e_1, \dots, e_n\}$ be the standard basis in $\mathbb{R}^n$ and $(e_i, e_j) = \delta_{ij}$. We let $L_n = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_n$ denote the standard n-dimensional lattice in $\mathbb{R}^n$.

Exercise 6.59. A_n $(n \geq 1)$: hyperplane $V \subseteq \mathbb{R}^{n+1}$, $V = (e_1 + e_2 + \cdots + e_{n+1})^{\perp}$,

$$R = \{\alpha \in V \cap L_{n+1} : (\alpha, \alpha) = 2\}.$$

- $R = \{e_i - e_j : i \neq j\}$;
- $S = \{e_i - e_{i+1} : i = 1, \dots, n\}$;
- $W = \mathrm{Aut}\{e_1, \dots, e_n, e_{n+1}\} = S_{n+1}$.

cf. roots for $\mathfrak{sl}_{n+1}(\mathbb{C})$.

Exercise 6.60. B_n $(n \geq 1)$: $V = \mathbb{R}^n$, $R = \{\alpha \in L_n : (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\}$.

- $R = \{\pm e_i, \pm e_1 \pm e_j, i \neq j\}$
- $S = \{e_1 - e_2, e_3 - e_2, \dots, e_n - e_{n-1}, e_n\}$.

$W = S_n \times (\mathbb{Z}_2)^n$.
$S_n \cong \mathrm{Aut}\{e_1, \dots, e_n\}$.
$(\mathbb{Z}_2)^n$: changing signs.
$A_1 \cong B_1$

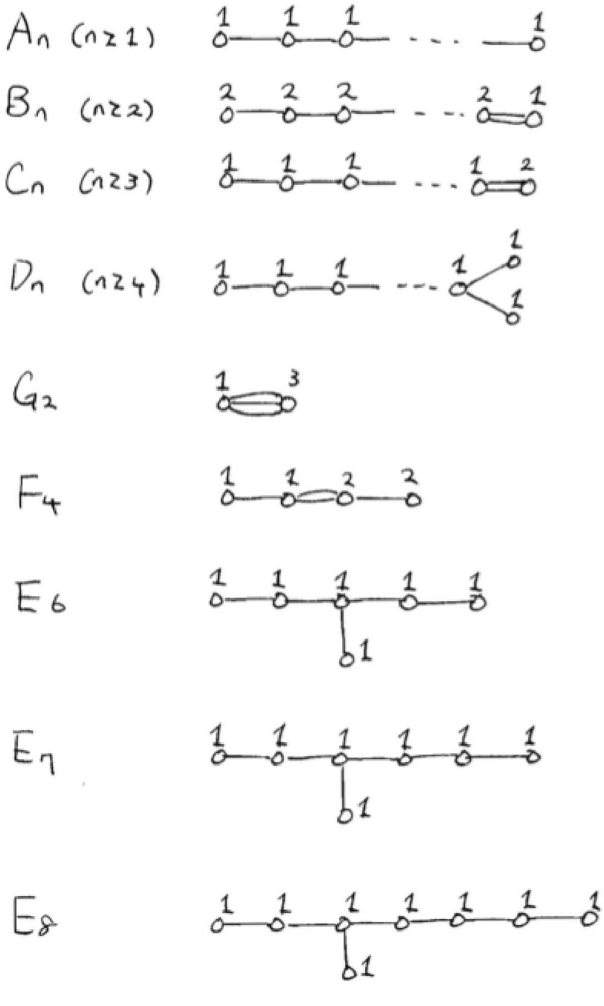


Figure 4. Classification of Dynkin diagrams

Exercise 6.61. C_n $(n \geq 1)$: Dual to B_n

- $R = \{\pm e_i \pm e_j, i \neq j, \pm 2e_i\}$.
- $S = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$.
- $W \cong S_n \times (\mathbb{Z}_2)^n$

(This is the so-called Langlands duality)

Exercise 6.62. D_n $(n \geq 2)$: $V = \mathbb{R}^n$ $R = \{\alpha \in L_n : (\alpha, \alpha) = 2\}$

- $R = \{\pm e_i \pm e_j : i \neq j$
- $S = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$
- $W \cong S_n \times (\mathbb{Z}_2^n)_0$

This means that changing sign in even number of $\{e_1, \ldots, e_n\}$

$D_2 \cong A_1 \times A_1$
$D_3 = A_3$.

Exercise 6.63. G_2: R was described; subset of integer algebraic numbers with norm 1 or 3 in field defined by $x^3 = 1$.

Exercise 6.64. Find W (dihedral group)

Exercise 6.65. F_4: $V = \mathbb{R}^4$, $L_4' = L_4 \cup \{\frac{e_1+e_2+e_3+e_4}{2}\}$, $R = \{\alpha \in L_4' : (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\}$

- $R = \{\pm e_i, \pm e_i \pm e_j, \text{ for } i \neq j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$
- $S = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$.

Exercise 6.66. E_8: $V = \mathbb{R}^8$, $L_8' = L_8 \cup \{\frac{1}{2}(e_1 + \cdots + e_8)\}$
$L_8' \supseteq L_8'' = \{\sum_{i=1}^{8} k_i e_I : \sum_{i=1}^{8} k_i \in 2\mathbb{Z}\}$.
$R = \{\alpha \in L_8'' : (\alpha, \alpha) = 2\}$.

- $R = \{\pm e_i \pm e_j (i \neq j), \frac{1}{2} \sum_{i=1}^{\infty} (-1)^{m_i} e_i, \sum m_i \in 2\mathbb{Z}\}$.
- $S = \{e_{i+1} - e_i, i = 1, 2, \ldots, 6; e_1 + e_2, \frac{1}{2}(e_1 + e_8 - \sum_{i=2}^{7} e_i)\}$.

$\mathbb{R}^6 \hookrightarrow \mathbb{R}^8$ $(e_7 = e_8 = 0)$
$\mathbb{R}^7 \subseteq \mathbb{R}^8$ $(e_8 = 0)$
Then for E_6: $R = (R \text{ for } E_8) \cap \mathbb{R}^6$
E_7: $R = (R \text{ for } E_8) \cap \mathbb{R}^7$.

8. Complex Root Systems

Let V be a finite-dimensional vector space over $\mathbb{C}$. For any nonzero $\alpha \in V$, we define s_α.

Definition. $R \subseteq V$ is a *complex root system* if R is finite, R does not contain 0, $\operatorname{span} R = V$, and

(i) For all $\alpha \in R$, there exists a unique $s_\alpha = 1 - \alpha^* \otimes \alpha$ such that $s_\alpha(R) = R$.
(ii) If α and $s_\alpha(\beta) - \beta$ are collinear, then $s_\alpha(\beta) - \beta = n\alpha$ for some $n \in \mathbb{Z}$.

Example. V_0 over $\mathbb{R}$, R a root system for V_0, then R complex root system for $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$. Expand s_α linearly.

The only example is as follows:

Theorem 6.67. *Let R be a complex root system for V over $\mathbb{C}$. Let $V = \operatorname{span}_{\mathbb{R}} R$. Then*

(a) R is a root system for V_0 over $\mathbb{R}$
(b) $V_0 \otimes_{\mathbb{R}} \mathbb{C} \cong V$.
(c) $s_\alpha : V \to V$ are linear expansions of $s_\alpha^0 : V_0 \to V_0$.

Sketch of proof. We have $s_\alpha(R) = R$ which implies $s_\alpha(V_0) = V_0$. Set $s_\alpha^0 = s_\alpha|_{V_0}$. For $\alpha, \beta \in R$, we have $s_\alpha^0(\beta) = \beta - \alpha^*(\beta)\alpha$, so that R is a root system for V_0. The rest of the proof is left as an exercise. $\square$

CHAPTER 7

Semisimple Complex Lie Algebras

1. Roots of Complex Lie Algebras

Let L be a finite-dimensional semisimple complex Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. For $\alpha \in \mathfrak{h}^*$, we define

$$L^\alpha = \{x \in L : [H, x] = \alpha(H)x \text{ for all } H \in \mathfrak{h}\}.$$

The elements of L^α are elements of weight α; $L^0 = \mathfrak{h}$ is abelian.

Definition. $\alpha \in \mathfrak{h}^*$ is a root of L if $\alpha \neq 0$ and $L^\alpha \neq \{0\}$.

The set $R_{\mathfrak{h}}$ of roots is commonly denoted simply by R.

We proved

$$L = \mathfrak{h} \oplus \sum_{\alpha \in R} L^\alpha.$$

Let $(\cdot, \cdot)$ be a Killing form on L.

Theorem 7.1. *The following holds:*

- *(i)* *L^α is orthonormal to L^β if $\alpha + \beta \neq 0$ and if $(\cdot,\cdot)|_{\mathfrak{h}}$ and $(\cdot,\cdot)|_{L^\alpha \oplus L^{-\alpha}}$ is nondegenerate.*
- *(ii)* *If $x \in L^\alpha$, $y \in L^{-\alpha}$, and $H \in \mathfrak{h}$, then $(H, [x, y]) = \alpha(H)(x, y)$.*
- *(iii)* *Let $x \in L^\alpha$, $\alpha \mapsto H_\alpha$ under $\mathfrak{h} \overset{\cong}{\longrightarrow} \mathfrak{h}^*$ defined by $(\cdot,\cdot)$. Then $[x, y] = (x, y)H_\alpha$ for $x \in L^\alpha$ and $y \in L^{-\alpha}$.*

Proof. (i) $(\cdot,\cdot)$ is invariant, so

$$([H, x], y) + (x, [H, y]) = (\alpha(H)x, y) + (x, \alpha(H)y) = 0$$

for each $x \in L^\alpha$ and every $y \in L^\beta$. Therefore, $(\alpha(H) + \beta(H))(x, y) = 0$. We may choose H so that $(\alpha + \beta)(H) \neq 0$, whereby we have $(x, y) = 0$.

So,

$$L = \mathfrak{h} \oplus \sum_{\alpha \in R} (L^\alpha \oplus L^{-\alpha}),$$

the decomposition into mutually orthogonal spaces. Then $(\cdot,\cdot)|_{\mathfrak{h}}$ and $(\cdot,\cdot)|_{L^\alpha \oplus L^{-\alpha}}$ are nondegenerate. This proves (i).

(ii) Since $(\cdot,\cdot)$ is invariant, $(H, [x, y]) = ([H, x], y)$ (why?). Therefore,

$$(H, [x, y]) = ([H, x], y) = \alpha(H)(x, y).$$

(iii) is left as an exercse. $\square$

We let

$$L = \mathfrak{h} \oplus \sum_{\alpha \in R} (L^\alpha \oplus L^{-\alpha})$$

and that L is a complex semisimple Lie algebra.

Theorem 7.2. *(a) R reduced complex root system.*

(b) If $\alpha \in R$, then L^α, $[L^\alpha, L^{-\alpha}]$ are one-dimensional. Furthermore, if $\mathfrak{h}_\alpha = [L^\alpha, L^{-\alpha}]$, then there exists a unique $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$.

(c) For all nonzero $e_\alpha \in L^\alpha$, there exists $f_\alpha \in L^\alpha$ such that $[e_\alpha, f_\alpha] = H_\alpha$, $[H_\alpha, e_\alpha] = 2e_\alpha$, and $[H_\alpha, f_\alpha] = -2f_\alpha$.

(d) If $\alpha, \beta \in R$ and $\alpha + \beta \neq 0$, then $[L^\alpha, L^\beta] = L^{\alpha+\beta}$.

This implies that L is a collection of $\mathfrak{sl}_2(\mathbb{C})$, the "elementary module". (d) tells us how we glue them together. To prove the theorem, we consider the properties of $\mathfrak{sl}_2(\mathbb{C})$ and the following proposition. Denote by $(\cdot,\cdot)$ the Killing form on L.

Proposition 7.3. *(a) If $\beta \neq -\alpha$, then $(L^\alpha, L^\beta) = 0$, and $(\cdot,\cdot)|_{\mathfrak{h}}$ and $(\cdot,\cdot)|_{L^\alpha \oplus L^{-\alpha}}$ are nondegenerate.*

(b) For all $x \in L^\alpha$, $y \in L^{-\alpha}$, $H \in \mathfrak{h}$ and $(H, [x, y]) = \alpha(H)(x, y)$

(c) Let $\alpha \in R$, $\alpha \mapsto H_\alpha$ under $\mathfrak{h} \cong \mathfrak{h}^*$ induced by $(\cdot,\cdot)$. Then $[x, y] = (x, y)H_\alpha$ for $x \in L^\alpha$, $y \in L^{-\alpha}$.

□

Proof. We shall break the proof into smaller steps.

Step 1. If $\alpha, \beta \in \mathfrak{h}^*$, then $[L^\alpha, L^\beta] \subseteq L^{\alpha+\beta}$. In fact, the Jacobi identity yields

$$[H, [x, y]] = [[H, x], y] + [x, [H, y]]$$

for all $x \in L^\alpha$ and $y \in L^\beta$. Therefore,

$$[H, [x, y]] = (\alpha + \beta)(x)[x, y],$$

and so $[x, y] \in L^{\alpha+\beta}$.

Step 2. $\operatorname{span} R = \mathfrak{h}^*$. If not, there exists $H \neq 0$ such that $H \in \mathfrak{h}$ and $\alpha(H) = 0$ for all $\alpha \in R$. If $\operatorname{span} R \neq \mathfrak{h}^*$, $(\operatorname{span} R)^\perp \subseteq \mathfrak{h}$. Take $H \in (\operatorname{span} R)^\perp$.

Now, $\alpha(H) = 0$ implies that $H \in Z(L)$. But L is semisimple, so $Z(L) = \{0\}$. This is a contradiction.

Step 3. If $\alpha \in R$, then $[L^\alpha, L^{-\alpha}]$ is one-dimensional. In fact, $[x, y] = (x, y)H_2$ by Theorem 7.1. Here $x \in L^\alpha$ and $y \in L^{-\alpha}$.

Step 4. If $\alpha \in R$, there exists a unique $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$. According to Step 3, it is enough to show that $\alpha|_{\mathfrak{h}_\alpha} \neq 0$. Assume the opposite, and choose $x \in L^\alpha$ and $y \in L^{-\alpha}$ such that $[x, y] \neq 0$. Set $z = [x, y]$. Then $\alpha(z) = 0$, which implies that $[z, x] = \alpha(z)x = 0$ and that $[z, y] = \alpha(z)y = 0$.

Therefore, $\mathfrak{a} = \operatorname{span}\{x, y, z\}$ such that $[x, y] = z$ and $[x, z] = [y, z] = 0$, which means $\mathfrak{a}$ is the Heisenberg algebra. In particular, it is nilpotent. By Lie's theorem, for all $\rho : \mathfrak{a} \to \operatorname{End} V$, where V is finite-dimensional, there exists a flag $\mathcal{F}$ invariant under $\rho(\mathfrak{a})$. Set $V = L$. Since $z \in [\mathfrak{a}, \mathfrak{a}]$, the map $\operatorname{ad} z : L \to L$ is nilpotent. But all nonzero elements of L are semisimple.

Step 5. Let $\alpha \in R$, $e_\alpha \neq 0$. There exists $f_\alpha \in L^{-\alpha}$ such that $[e_\alpha, f_\alpha] = H_\alpha$. In fact, $(\cdot,\cdot)$ defines duality for L^α and $L^{-\alpha}$. Therefore there exists $y \in L^{-\alpha}$ such that $(e_\alpha, y) \neq 0$. By Theorem 7.1, $[e_\alpha, y] \neq 0$. By scaling y, we get f_α.

Summary thus far. $\operatorname{span}\{e_\alpha, f_\alpha, H_\alpha\} \cong \mathfrak{sl}_2(\mathbb{C})$. We consider L as module over $\mathfrak{sl}_2(\mathbb{C})$.

Step 6. If $\alpha \in R$, then $\dim L^\alpha = 1$. Assume that $\dim L^\alpha > 1$. L^α and $L^{-\alpha}$ are dual to each other. If $\dim L^\alpha > 1$, there exists $z \in L^{-\alpha}$: $(e_\alpha, z) = 0$. so $[e_\alpha, z] = 0$ by Theorem 7.1.

On the other hand, $[H_\alpha, z] = -\alpha(H_\alpha)z$. So z is a primitive element with negative weight. But, for all finite-dimensional $\mathfrak{sl}_2$-modules generated by a primitive element, the elements have positive weight. This is a contradiction, and so $\dim L^\alpha = 1$.

Step 7. We note that $\mathfrak{h}_\alpha \oplus L^\alpha \oplus L^{-\alpha} \cong \mathfrak{sl}_2(\mathbb{C})$.

Step 8. f_α is unique, for $\dim L^{-\alpha} = 1$.

Step 9. We claim that if $\alpha, \beta \in R$, then $\beta(H_\alpha) \in \mathbb{Z}$ and $\beta - \beta(H_\alpha)\alpha \in R$. Therefore, R is a root system.

To see this, we let $y \in L^\beta$ and $y \neq 0$, set $p = \beta(H_\alpha)$. Then $[H_\alpha, y] = \beta(H_\alpha)y = py$. Consider L as $\mathfrak{sl}_2$-module. y has a weight. All $\mathfrak{sl}_2$-weights are integers, then p is integer.

If $p \geq 0$, set $z = f_\alpha^p y$. If $p < 0$, set $z = e_\alpha^{-p} y$. It shows that $\beta - p\alpha$ is a root.

Step 10 (Summary thus far). R is a root system. H_α dual to α. In fact, Step 2 implies that $\operatorname{span} R = \mathfrak{h}^*$. If $\alpha \in R$, set $s_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$ for all β. Then s_α is a reflection. Since $s_\alpha(\beta) \in R$ by Step 9, R is a root system.

Step 11. R is a reduced root system. Assume that $\alpha \in R$ and $2\alpha \in R$. Let y be a nonzero element of $L^{2\alpha}$. Then $[H_\alpha, y] = 2H_\alpha(\alpha)y = 4y$. We note that

$$[e_\alpha, y] \in L^{\alpha+2\alpha} = L^{3\alpha} = \{0\},$$

so $[e_\alpha, y] = 0$. Then $[H_\alpha, y] = [[e_\alpha, f_\alpha], y] = [e_\alpha, [f_\alpha, y]]$ by the Jacobi identity (why?) Now, $[f_\alpha, y] \in L^{-\alpha+2\alpha} = L^\alpha$. But any element in L^α is collinear to e_α, for $\dim L^\alpha = 1$. So $[e_\alpha, [f_\alpha, y]] = 0 = [H_\alpha, y] = 4y$, a contradiction.

Step 12. Let α, β be noncollinear,

$$\begin{aligned} p &= \max\{n : \beta - n\alpha \text{ is a root.}\}, \\ q &= \max\{m : \beta + m\alpha \text{ is a root.}\}. \end{aligned}$$

Set

$$E = \sum_{k \in \mathbb{Z}} L^{\beta + k\alpha}.$$

We claim that $\dim E = p + q + 1$. Indeed, $\beta(H_\alpha) = p - q$ and $\operatorname{ad}(e_\alpha) : L^{\beta+k\alpha} \cong L^{\beta+(k+1)\alpha}$ for $-p \leq k \leq q - 1$ (why?).

In fact, E is an $\mathfrak{sl}_2$-module. Its weights are $\beta(H_\alpha)+2k$ (Why?). E is irreducible, and $\dim E = m + 1$, where $m = \beta(H_\alpha) + 2q$ (why?) It follows that $\operatorname{ad} e_\alpha$ is an isomorphism.

Step 13. If $\alpha, \beta \in R$ and $\alpha + \beta \in R$, then $[L^\alpha, L^\beta] = L^{\alpha+\beta}$. (Why? Use Step 12) □

2. Borel Subalgebras of Complex Lie Algebras

Let L be a semisimple complex Lie algebra. We know that

$$L = \mathfrak{h} \oplus \sum_{\alpha \in R} (L^\alpha \oplus L^{-\alpha}),$$

where $\mathfrak{h}$ is a Cartan subalgebra. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of R and write $R = R_+ \cup R_-$. Set $\mathfrak{n}_+ = \sum_{\alpha \in R_+}$, $\mathfrak{n}_- = \sum_{\alpha \in R_+} L^{-\alpha}$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, the Borel subalgebra. Then

$$L = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Theorem 7.4. *(a)* $\mathfrak{n}_+$ *and* $\mathfrak{n}_-$ *consist of nilpotent elements (whereby they are nilpotent).*

(b) $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$*. Therefore,* $\mathfrak{b}$ *is solvable.*

PROOF. (a) Let $x \in \mathfrak{n}_+$, $k \in \mathbb{N}$ and $\beta \in \mathfrak{h}^*$. Then $(\operatorname{ad} x)^k L^\beta \subseteq L^{\beta+\alpha_1+\cdots+\alpha_k}$, where each a_i is in R_+. Note that

$$\beta + \alpha_1 + \cdots + \alpha_k \notin R \cup \{0\}$$

for $0 \ll k$. Since $\operatorname{ad} x$ is nilpotent, $\mathfrak{n}_+$ is nilpotent. Similarly for $\mathfrak{n}_-$.

(b) Enough to note that $[\mathfrak{h}, \mathfrak{n}_+] = \mathfrak{n}_+$. The rest is left as an exercise. ☐

Recall that $\mathfrak{b}$ depends on $\mathfrak{h}$ and S.

Theorem 7.5 (Borel-Morozov)**.** *Let* L *be a semisimple Lie algebra. Then for any solvable Lie subalgebra* $\mathfrak{a} \subseteq L$*, there exists an automorphism* $\varphi : L \to L$ *such that* $\varphi(\mathfrak{a})$ *is in the Borel subalgebra. In particular,* $\mathfrak{b}$ *is the maximal solvable subalgebra of* L*.* ☐

3. Generators and Relations for Semisimple Complex Lie algebras

Let $S = \{\alpha_1, \ldots, \alpha_n\}$ a basis in R, and R a root system for L. Then $n = \dim \mathfrak{h} = \operatorname{rank} L$. We write H_i, e_i, and f_i to denote H_{α_i} in $\mathfrak{h}$, e_{a_i} in L^{α_i}, and f_{α_i} in $L^{-\alpha_i}$, respectively. We choose e_i and f_i such that $[e_i, f_i] = H_i$, and set $n(i,j) = \alpha_j(H_i)$. Recall that $(n(i,j))$ is the Cartan matrix.

Theorem 7.6. *(a)* $\mathfrak{n}_+$ *is generated by* e_i*,* $\mathfrak{n}_-$ *is generated by* f_i*, and* L *is generated by* e_i*,* f_i*, and* H_i*.*

(b) For all $i \neq j$*, we have the following* Weyl relations*:*
- $[H_i, H_j] = 0$,
- $[e_i, f_i] = H_i$,
- $[e_i, f_j] = 0$,
- $[H_i, e_j] = n(i,j)e_j$,
- $[H_i, f_j] = -n(i,j)f_j$.

(c) For all $i \neq j$*, we have the following* Serre relations*:*
- $(\operatorname{ad} e_i)^{-n(i,j)+1}(e_j) = 0$,
- $(\operatorname{ad} f_i)^{-n(i,j)+1}(f_j) = 0$.

PROOF. (a) Recall that any root $\alpha \in R$ can be written as

$$\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$$

and any sum $\alpha_{i_1} + \cdots + \alpha_{i_m}$ is a root if $m \leq k$. Therefore, the collection of

$$e_\alpha = [e_{i_k}, [e_{i_{k-1}}, \ldots, [e_{i_2}, e_{i_1}]] \ldots]$$

is a linear basis in $\mathfrak{n}_+$ and so $\mathfrak{n}_+$ is generated by $e_1, \ldots, e_n$. Similarly for $\mathfrak{n}_-$ and L.

(b) We shall only prove $[e_i, f_j] = 0$ if $i \neq j$. If $[e_i, f_j] \neq 0$, its weight is $\alpha_i - \alpha_j$. No such weights ($R = R_+ \cup R_-$).

(c) Set

$$\begin{aligned} \Theta_{ij}^+ &= (\operatorname{ad} e_i)^{-n(i,j)+1}(e_j) \\ \Theta_{ij}^- &= (\operatorname{ad} f_i)^{-n(i,j)+1}(f_j). \end{aligned}$$

The weight of Θ_{ij}^+ is

$$\alpha_j - n(i,j)\alpha_i + \alpha_i = s_{\alpha_i}(\alpha_j - \alpha_i).$$

Since there is no such root, $\Theta_{ij}^+ = 0$. Similarly, $\Theta_{ij}^- = 0$. $\square$

Theorem 7.7. *Lie algebra* $\mathfrak{n}_+$ *is defined by generators* $e_1, \ldots, e_n$ *and relations* θ_{ij}^+. $\square$

Let $\mathfrak{F}_n$ be the free Lie algebra with generators $e_1, \ldots, e_n$ and I the ideal generated by Θ_{ij}^+. We take the epimorphism $\mathfrak{F}_n \to \mathfrak{n}_+$ defined by $e_i \mapsto e_i$. Let I be the kernel.

Similarly,

Theorem 7.8. *L is defined by generators (e_i), (f_i), and (H_i) satisfying the Weyl and Serre relations.* $\square$

Example. Type G_2. Generators for $\mathfrak{n}_+$ are e_1 and e_2, and the relations are $[e_1, [e_1, e_2]] = 0$ (namely $n(1,2) = -1$) and $[e_2, [e_2, [e_2, [e_2, e_1]]]] = 0$ (namely $n(2,1) = -3$).

Remark. Monomial-type Serre relations come mostly from $\mathfrak{n}_+$ for semisimple Lie algebras.

Example. The *Witt algebra* W_1 is an infinite-dimensional Lie algebra generated by the linear basis $(e_i)_{i \in \mathbb{Z}}$ and the relations $[e_i, e_j] = (j - i)e_{i+j}$.

Exercise 7.9. Check the Jacobi identity.

Nilpotent part: Linear basis $(e_i)_{i \geq 1}$. There are two generators, namely e_1 and e_2. Relations:

$$[e_1, [e_1, [e_1, e_2]]] = 6e_5.$$
$$[e_2, [e_2, e_1]] = -e_5.$$

Relation:

$$(\operatorname{ad} e_1)^3(e_2) + b(\operatorname{ad} e_2)^2(e_1) = 0.$$

This is not a monomial.

Exercise 7.10. Compute e_7 in two different ways. This gives another relation. There are no other relation.

Here $\mathfrak{h} = \mathbb{C}e_0$.

Generators (e_i), (f_i), and (H_i) for a semisimple Lie algebra define a linear map $\varphi : L \to L$ such that $\varphi(e_i) = -f_i$, $\varphi(f_i) = -e_i$, and $\varphi(H_i) = -H_i$.

Exercise 7.11. φ is an automorphism and $\varphi^2 = \mathrm{id}$.

Exercise 7.12. Define φ for $\mathfrak{sl}_n(\mathbb{C})$.

4. Roots and Semisimple Complex Lie Algebras

Let L_1 and L_2 be complex Lie algebras and R_1 and R_2 their root systems, respectively.

Theorem 7.13. *If $R_1 \cong R_2$, then $L_1 \cong L_2$.*

Let us consider the construction first. Let S_1 and S_2 be bases for R_1 and R_2, respectively. We assume that $r : S_1 \xrightarrow{\cong} S_2$ is a bijection such that $n(r(\alpha), r(\beta)) = n(\alpha, \beta)$ for all $\alpha, \beta \in S_1$.

For all $\gamma \in S_1$ and $\delta \in S_2$, we choose the corresponding $e_\gamma^1 \in L_1$ and $e_\delta^2 \in L_2$. Then there exists a unique $f : L_1 \to L_2$ such that $f(e_\gamma^1) = e_\delta^2$. We leave it as an exercise for the reader to finish the proof.

Theorem 7.14 (Summary)**.** *Let R be a reduced complex root system. There exists a semisimple Lie algebra L such that R is a root system for L. And L is defined uniquely up to an isomorphism.*

Construction: Choose $S = \{\alpha_1, \dots, \alpha_n\}$ basis in R. Construct L by generators (e_i), (f_i), and (H_i) satisfying the Weyl and Serre relations. Then prove that L is finite-dimensional semisimple Lie algebra with root system R.

Exercise 7.15. L is simple if and only if R is irreducible as a root system.

How do we prove that L is semisimple and finite-dimensional? We construct Lie algebra $\mathfrak{a}$ defined by the generators and Weyl relations: $\mathfrak{a} = F \oplus \mathfrak{h} \oplus E$, where F is generated by $f_1, \dots, f_n$, E is generated by E is generated by $e_1, \dots, e_n$, and $\mathfrak{h}$ is spanned by $\{H_i\}$. Note that $\Theta_{ij}^{+} \in E$ and $\Theta_{ij}^{-} \in F$. Let u^+ and u^- be ideals in E and F, respectively, generated by Θ_{ij}^{+} and Θ_{ij}^{-}, respectively. Set $u = u^+ \oplus u^-$.

Step 1. Check that u^+, u^- and u are ideals in $\mathfrak{a}$.

Step 2. Set $L = \mathfrak{a}/u$. Then $L = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+ = E/u^+$ and $\mathfrak{n}_- = F/u^-$.

Step 3. $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ (Weyl relations) are locally nilpotent in L, viz., for all $z \in L$, there exists $k \in \mathbb{N}$ such that $(\operatorname{ad} e_i)^k z = 0$ (same for $\operatorname{ad} f_i$).

Step 4. Show that $\dim L^\alpha = 1$.

Step 5. $\dim L = \dim \mathfrak{h} + |R|$.

Step 6. L is semisimple.

5. Weights and Semisimple Complex Lie Algebras

Let L be a complex semisimple Lie algebra, R a root system, and $S = \{\alpha_1, \dots, \alpha_n\}$ a basis. Recall that $L = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, $\mathfrak{n}_- = \sum_{\alpha \in R} L^\alpha$, $\mathfrak{n}_+ = \sum_{\alpha \in R} L^\alpha$, $\dim L^\alpha = 1$, $e_\alpha \in L^\alpha$, $f_\alpha \in L^{-\alpha}$, $\alpha \in R_+$, and $[e_\alpha, f_\alpha] = H_\alpha \in \mathfrak{h}$.

Let V be an L-module, not necessarily finite-dimensional. For $w \in \mathfrak{h}^*$, we set $V^w = \{v \in V : Hv = w(H)v\}$. The elements of V^w have weight w, and $\dim V^w$ is the multiplicity of w. w is a weight if $V^w \neq \{0\}$.

We are going to study structure V using representations of $\mathfrak{sl}_2(\mathbb{C})$.

Proposition 7.16. *(a) Let $w \in \mathfrak{h}^*$ and $\alpha \in R$. Then $L^\alpha V^w \subseteq V^{w+\alpha}$.*
(b) Set $V' = \sum_{w \in \mathfrak{h}^*} V^w$. Then V' is a direct sum and V' is an L-submodule.

Proof. (a) Observe that

$$H(e_\alpha v) = e_\alpha(Hv) + [H, e_\alpha]v = e_\alpha(Hv) + \alpha(H)e_\alpha.$$

If $v \in V^w$, then $Hv = w(H)v$. So

$$H(e_\alpha v) = w(H)e_\alpha v + \alpha(H)e_\alpha v = (w(H) + \alpha(H))e_\alpha v/$$

This proves (a).

(b) V' is a direct sum (Linear algebra). (a) shows that V' is an L-submodule. □

Definition. $v \in V$ is *primitive* of weight w if $v \neq 0$ for all $\alpha \in R_+$ (or S), we have $e_\alpha v = 0$ and $v \in V^w$.

In other words, v is an eigenvector for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$.

Proposition 7.17. *Let $v \in V$ be a primitive of weight w, E submodule of V generated by v. Then*

(1) $E = \operatorname{span}\{f_{\beta_1}^{m_1} \cdots f_{\beta_k}^{m_k} v, \beta_1, \dots, \beta_k \in R_+, m_i \in \mathbb{N}\}$.
(2) Weights of E are $w - \sum_{i=1}^n p_i\alpha_i$, where $p_i \in \mathbb{N}$.
(3) $\dim E^w = 1$.
(4) E is indecomposable.

Proof. (1) $A \bigcup L$, $B = \bigcup \mathfrak{b}$, and $C = \bigcup \mathfrak{n}_-$. Now, $L = \mathfrak{n}_- \oplus \mathfrak{b}$ implies that $A = C \otimes B$. Then $E = Av = CBv$. However, $Bv = \mathbb{C}v$ because v is eigenvector for $\mathfrak{b}$. Therefore, $E = Cv$. By the Poincaré-Birkhoff-Witt theorem,

$$\left\{f_{\beta_1}^{m_1} \cdots f_{\beta_k}^{m_k}\right\}$$

is a basis in C. This proves (1).

(2) By Proposition 7.16, $f_{\beta_1}^{m_1} \cdots f_{\beta_k}^{m_k} v$ has weight $w - \sum_{i=1}^k m_i\beta_i$. Each

$$\beta_i = \sum_{j=1}^n q_{ij}\alpha_j$$

with $q_{ij} \in \mathbb{Z}_{\geq 0}$. This proves (2).

The proof of (3) is left as an exercise.

(4) If $E = E_1 \oplus E_2$, then $E^w = E_1^w \oplus E_2^w$ and $\dim E^w = 1$, so $E^w = E_1^w$ or $E^w = E_2^w$, then $E = E_1$ or $E = E_2$. □

Theorem 7.18. *Let V be an irreducible L-module, and $v \in W$ a primitive of weight w. Then*

(a) v is, up to a scalar multiple, the unique primitive element of V.
(b) Any weight of V can be expressed as $\pi = w - \sum_{m_i \in \mathbb{N}} m_i d_i$.
(c) If irreducible V_i has primitive elements of weight w_i for $i = 1, 2$, then $V_1 \cong V_2$ if and only if $w_1 = w_2$.

Definition. In this case, V is the module with *highest weight* w.

Proof. (b) Let E be submodule generated by v. Then $E = V$ because V is irreducible. THen use Proposition 7.17.

(a) Let v' be a primitive element with weight w'. By (b), we have

$$\begin{aligned} w' &= w - \sum m_i\alpha_i \\ w &= w' - \sum m_i'\alpha_i, \end{aligned}$$

where m_i and m_i' are nonnegative. Then

$$0 = \sum (m_i + m_i')\alpha_i,$$

which then implies $m_i + m_i' = 0$ for all i. Therefojre, $m_i = m_i' = 0$, and so $w = w'$. We know that $\dim E^w = 1$.

(c) Let v_i be a primitive element for irreducible V_i with highest weight w for $i = 1, 2$. Set $V = V_1 \oplus V_2$. Then $v = v_1 + v_2$ is a primitive element with weight w. Let E be a submodule of V generated by v. We use projection $\pi_1 : V \to V_1$. Then $\pi_1|_E : E \to V_1$ is a surjective homomorphism. The kernel is $N_1 = V_2 \cap E$ (here we assume that $V_2 \hookrightarrow V$). Note that W_1 is a submodule in V_2 and that $v_2 \notin N_1$

(why?). But V_2 is irreducible and that $N_1 \subsetneq V_2$, whence $N_1 = \{0\}$. So $E \xrightarrow{\cong} V_1$. So, $V_1 \xrightarrow{\cong} V_2$. □

Remark. We will show that any finite-dimensional irreducible module is a highest weight module. Not true for infinite-dimensional irreducible modules.

Theorem 7.19. *For all $w \in \mathfrak{h}^*$, there exists a unique irreducible L-module with highest weight w.*

Proof. Let $L_w = \mathbb{C}v$. This is a $\mathfrak{b}$-module such that $Hv = w(H)v$ for all $H \in \mathfrak{h}$ and $e_\alpha v = 0$ for all $\alpha \in R_+$. Now, L_w is a $U\mathfrak{b}$-module. Take *induced* UL-modules

$$V_w = UL \otimes_{U\mathfrak{b}} L_w$$

where $\mathfrak{b} \subseteq L$ and $U\mathfrak{b} \subseteq UL$.

Now, V_w is generated by $1 \otimes v$. By an abuse of notation, we write $v = 1 \otimes v$. I do not claim that V_w is irreducible. Set

$$W = \sum_{\pi \neq w} V_w^\pi.$$

If U is an L-submodule and $U \neq V$, then $U \subseteq W$ (why?)

Let N_w submodule generated by all submodules U. Then $N_w \subseteq W$, and so $N_w \neq V_w$. Set $E_w = V_w/N_w$. We claim that E_w is irreducible of highest weight w (why?). □

Summary: There is a one-to-one correspondence between $\mathfrak{h}^*$ and modules with highest weights.

We now consider finite-dimensional irreducible modules

Theorem 7.20. *Let $w \in \mathfrak{h}^*$, E_w irreducible module with highest weight w. Then E_w is finite-dimensional if and only if $w(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R_+$.* □

Proposition 7.21. *Let $V \neq \{0\}$ be a finite-dimensional L-module. Then*
(a) $V = \bigoplus V^\pi$, where $\pi \in R$.
(b) $\pi(H_\alpha) \in \mathbb{Z}$ for all α
(c) V contains a primitive element.
(d) If V is generated by primitive elements, then V is irreducible.

Proof. (a) Any $H \in \mathfrak{h}$ is semisimple, $\operatorname{ad} H$ is semisimple. Since all elements from $\mathfrak{h}$ commute, all $\operatorname{ad} H$ is diagonalizable in same basis (Linear algebra)

(c) V is $\mathfrak{b}$-module, $\mathfrak{b}$ solvable. Use Lie's theorem.

(d) By Proposition 7.17, V is indecomposable. By Weyl's theorem, V is semisimple. It follows that V is irreducible. □

Corollary 7.22. *Any irreducible finite-dimensional L-module is highest-weight module.*

We set $H_i = H_{\alpha_i}$, $S_i = S_{\alpha_i}$.

Theorem 7.23. *Let $w \in \mathfrak{h}^*$, E_w irreducible L-module with highest weight w. Then E_w is finite-dimensional if and only if $w(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R_+$.*

Proof. If v is primitive, E_w is finite-dimensional, then consider E_w as $\mathfrak{sl}_2(\mathbb{C})$ module over $\operatorname{span}\{e_i, f_i, H_i\}$. Then $w(H_{\alpha_i}) \in \mathbb{Z}_{\geq 0}$. If $\alpha \in R$, then $\alpha = \sum p_i \alpha_i$, $p_i \in \mathbb{Z}_{\geq 0}$. Conversely:

Step 1. E_w is a sum of $\mathfrak{a}_i$-modules of finite dimension, where $\mathfrak{a}_i = \operatorname{span}\{e_i, f_i, H_i\}$. Let $v \in E_w$ be a primitive element. Set

- $m_i = w(H_i)$, $i = 1, \ldots, n$
- $v_i = f_i^{m_{i+1}} v$, $i = 1, \ldots, n$.

If $j \neq i$, then e_j and f_i commute, so

$$e_i v_i = f_i^{m_i} e_j v = 0,$$

for v is primitive. Note that $e_i v_i = 0$ (why?)

So, if $v_i \neq 0$, then v_i is primitive with weight $w - (m_i + 1)\alpha_i$. But there exists primitive element v, unique up to a scaling. Since $w \neq w - (m_i + 1)\alpha_i$, we have $v_i = 0$.

Therefore,

$$\operatorname{span}\{f_i^p v : 0 \leq o \leq m_i\}$$

is finite-dimensional $\mathfrak{a}_i$-module. Then E_i is L-submodule of E_w (why?) E_w is irreducible, so $E_w = E_i$.

Step 2. Let P_w be a set of all weights of E_w. Then $s_i(P_w) \subseteq P_w$. In fact, let $\pi \in P_w$, $y \in E_w^\pi$, $y \neq 0$. Then $p_i = \pi(H_i) \in \mathbb{Z}_{\geq 0}$. Set

$$x = \begin{cases} f_i^{p_i} y & \text{if } p_i \geq 0; \\ f_i^{-p_i} y & \text{if } p_i < 0. \end{cases}$$

Note that $x \neq 0$ (why? Use the $\mathfrak{sl}_2$-property). The weight of x is

$$\pi - p_i \alpha_i = \pi - \pi(H_i)\alpha_i = s_i(\pi).$$

So, there exists an element of weight $s_i(\pi)$, and so $s_i(P_w) \subseteq P_w$.

Step 3. P_w is finite (nontrivial). Let $\pi \in P_w$. Then $\pi = w - \sum p_i \alpha_i$, where $p_i \in \mathbb{N}$. It is enough to show that $p_1, \ldots, p_n \leq N$ for some N. Let $-S = \{-\alpha_1, \ldots, -\alpha_n\}$ be a basis. As we proved, there exists a unique element $w \in W$ such that $w(S) = -S$. We know that w is product of reflections s_i, so $w(\pi) \in P_w$. Also, $w(\pi) = w - \sum q_i \alpha_i$. Apply w^{-1}, we get

$$\pi = w^{-1}(\pi) + \sum r_i \alpha_i,$$

where $r_i \geq 0$. It follows that

$$w - w^{-1}(\pi) = \sum (p_i + r_i)\alpha_i.$$

The left-hand side depends on our choice of s. There are finite number of choices. There are finite set of sums $p_i + r_i$. So, $p_i + r_i < N$ for some N. So $p_i < N$, $i = 1, \ldots, n$.

Step 4. E_w has a finite number of weights (Step 3). Each of these weights has finite multiplicity. So, $\dim E_i < \infty$ and $E_i = E_w$. □

Exercise 7.24. The set of weights P_w is invariant under action of w. In fact, π and $w(\pi)$ have the same multiplicity.

Remark. For basis $\{H_1, \ldots, H_n\}$ in $\mathfrak{h}$, we introduce a *biorthogonal basis* $\{w_1, \ldots, w_n\}$ in $\mathfrak{h}^*$ by setting $w_i(H_j) = \delta_{ij}$. We call $w_1, \ldots, w_n$ the *fundamental weights* of (R, S). So, any weight of E_w is a linear combination of fundamental weights with coefficients from $\mathbb{Z}_{\geq 0}$.

Example. $L = \mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{h} = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) : \sum \lambda_i = 0\}$.

Roots: $\alpha_{ij}(H) = \lambda_i - \lambda_j$, where $H = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$ for all $i \neq j$.

Basis: $\alpha_i = \alpha_{i,i+1}$ for all $i = 1, \ldots, n$, where $H_i = \mathrm{diag}(0, \ldots, 1, -1, 0, \ldots,)$ and $\alpha_i(H_i) = 2$; here the 1 is on the ith diagonal.

Fundamental weights: $w_i(H_i) = \lambda_1 + \cdots + \lambda_i$ for $i = 1, \ldots, n$.

Exercise 7.25. $E_{w_i} = \wedge^i \mathbb{C}^{n+1}$, $E_{w_1} = \mathbb{C}^{n+1}$ the standard representation.

6. Action of Weyl group

Proposition 7.26. *Let R is a root system. Weyl group W acts on set of bases for R transitively and without fixed points.*

Proof. Transitivity was proved. Fix S basis. Let w find weight for S. $w \mapsto E_w$ irreducible finite-dimensional L-module. Assume $w(S) = S$. Then $w^{-1}(w)$ is a weight for E_w (Remark 1). So $w(w) = w + \sum p_i' \alpha_i$ cannot be true unless all $p_i' = 0$ (because w is highest weight).

So $w(w) = w$. Since fundamental weights form a basis for $\mathfrak{h}^*$, we have $w =$ id. $\square$

7. Characters

Let L be a complex semisimple Lie algebra, R a root system, and $S = \{\alpha_1, \ldots, \alpha_n\}$ a basis. We define $P \subseteq \mathfrak{h}^*$ by

$$P = \{\pi \in \mathfrak{h}^* : \pi(H_\alpha) \in \mathbb{Z}, \alpha \in R\}.$$

Note that P is a free abelian group with the fundamental weights $\{w_1, \ldots, w_n\}$ as a basis.

We define a group ring $A = \mathbb{Z}[P]$. A is $\mathbb{Z}$-algebra with basis e^π for all $\pi \in P$. Multiplication works as follows: $e^{\pi_1} \cdot e^{\pi_2} = e^{\pi_1 + \pi_2}$. Let V be a finite-dimensional L-module, π a weight of V of multiplicity m_π, where $(m_\pi = \dim V^\pi$.

Definition. The *character* of V is

$$\mathrm{ch}(V) = \sum m_\pi e^\pi \in \mathbb{Z}[P],$$

where $m_\pi = \dim V^\pi$

Proposition 7.27. *(a)* $\mathrm{ch}(V)$ *is W-invariant, where W is the Weyl group.*
(b) $\mathrm{ch}(V \oplus V') = \mathrm{ch}(V) + \mathrm{ch}(V')$ *and* $\mathrm{ch}(V \otimes V') = \mathrm{ch}(V)\,\mathrm{ch}(V')$.
(c) $V \cong V'$ *if and only if* $\mathrm{ch}\, V = \mathrm{ch}\, V'$.

Proof. We prove (c) by induction on $\dim V$. Note that $\dim(V) = 0$ if and only if $\mathrm{ch}(V) = 0$. We now let $\dim V > 0$. If $V \cong V'$, then $\mathrm{ch}\, V = \mathrm{ch}\, V'$. Let $\mathrm{ch}\, V = \mathrm{ch}\, V'$. This implies $P_V = P_{V'}$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a basis. Choose $w \in P_V$ such that $w + \alpha_i \notin P_V$ for all i (why can we do it?). Let $v \in V$ such that $v \neq 0$. Then v is primitive (why?). Then V_1, a submodule of V generated by v, is irreducible with highest weight w. Since V is semisimple (why?), $V = V_1 \oplus V_2$. Similarly, $V' = V_1' \oplus V_2'$, where V_1' is irreducible with highest weight w. It follows that $V_1 \cong V_1'$, whence $\mathrm{ch}\, V_1 = \mathrm{ch}\, V_1'$ (by b). By induction $V_2 \cong V_2'$, and so $V \cong V'$.

Now, W acts on P, therefore W acts on $\mathbb{Z}[P]$. Denote by $\mathbb{Z}[P]^W$ the space of invariants. Let $T_i = \mathrm{ch}(V_{w_i})$, where w_i is the ith fundamental weight. Then

$$\mathbb{Z}[P]^W \cong \mathbb{Z}[T_1, \ldots, T_n].$$

In particular, $T_1, \ldots, T_n$ are algebraically independent. $\square$

Recall the *Grothendieck group* (from K-theory). Let $\mathcal{C}$ be an abelian category and define the abelian group $\mathrm{Gr}(\mathcal{C})$ by the generators $[X]$, where $X \in \mathrm{ob}(\mathcal{C})$ and the relations defined as follows: if

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence, then $[Y] = [X] + [Z]$.

Theorem 7.28. *Let $\mathcal{C}$ be the category of finite-dimensional L-modules, where L is a complex semisimple Lie algebra of finite dimension. Then*

$$\mathrm{ch} : \mathrm{Gr}(\mathcal{C}) \xrightarrow{\cong} \mathbb{Z}[P]^W$$

and $\mathrm{ch}\, Y = \mathrm{ch}\, X + \mathrm{ch}\, Z$. □

How do we compute characters for finite-dimensional irreducible L-modules? Let us introduce some notations.

(1) W acts on $\mathfrak{h}^*$ and $w \in W$. Define $\varepsilon(w) = \det w$. Note that $\varepsilon(w) = 1$ if w is a product of even number of reflections, and $\varepsilon(w) = -1$ if w is a product of odd number of reflections.

(2) Recall that

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Then $\rho(H_{\alpha_i}) = 1$ for all $\alpha_i \in S$, so $\rho \in P$.

(3) Set

$$D = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}),$$

which is in $\mathbb{Z}\left[\frac{1}{2}P\right]$. In fact,

$$E = \sum_{w \in W} \varepsilon(w) e^{w(\rho)},$$

which is in $\mathbb{Z}[P]$.

Theorem 7.29 (Weyl character formula)**.** *Let V_w irreducible highest weight module. Then*

$$\mathrm{ch}(V_w) = \frac{1}{D} \sum_{w \in W} \varepsilon(w) e^{w(w+\rho)}.$$

□

The above theorem was proved by H. Weyl in 1922 using the theory of compact Lie groups. The algebraic proof was given in 1954, but it is not very natural.

Example. Let

- $L = \mathfrak{sl}_2(\mathbb{C})$
- $R_+ = \{\alpha = 2\rho\}$
- $P = \{n\rho\}_{n \in \mathbb{Z}}$
- $w = m\rho$, $m \geq 0$.
- $W = \{\mathrm{id}, s_\alpha\}$.

Then, for $w = \mathrm{id}$,

$$\begin{aligned}\mathrm{ch}(V_w) &= \frac{e^{(m+1)\rho} - e^{-(m+1)\rho}}{e^{\rho} - e^{-\rho}} \\ &= e^{m\rho} + e^{(m-2)\rho} + \cdots + e^{-m\rho} \\ &= \sum_{\pi \in P} m_\pi e^{\pi}.\end{aligned}$$

Corollary 7.30.

$$\dim V_w = \prod_{\alpha \in R_+} \frac{\langle w+\rho, H_\alpha\rangle}{\langle \rho, H_\alpha\rangle} = \prod_{\alpha \in R_+} \frac{(w+\rho, \alpha)}{(\rho, \alpha)}.$$

□

Let V be a finite-dimensional L-module (not necessarily irreducible). Let E_w be the irreducible highest-weight submodule of V. Let $n(V, w)$ be the multiplicity of E_w in V[1]

Corollary 7.31. *$n(V, w)$ equals the coefficient for $e^{w+\rho}$ in $D\,\mathrm{ch}(V)$.* □

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[1] For $V = \oplus$ irreducible submodules, $n(V, w)$ is the number of these submodules isomorphic to E_w.

CHAPTER 8

Classical Lie Algebras

1. Symplectic Lie Algebra

Definition. Let V be a $2n$-dimensional vector space over $\mathbb{C}$ with a nondegenerate skew-symmetric bilinear form $Q : V \times V \to \mathbb{C}$. The *symplectic Lie algebra* is defined to be the collection

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{A \in \operatorname{End} V : Q(Ax, y) + Q(x, Ay) = 0\}.$$

Exercise 8.1. $\mathfrak{sp}_{2n}(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

We now derive the matrix presentation of $\mathfrak{sp}_{2n}(\mathbb{C})$. Choose basis $e_1, \ldots, e_{2n}$ in V such that

$$\begin{aligned} Q(e_i, e_{i+n}) &= 1 \\ Q(e_{i+n}, e_i) &= -1 \\ Q(e_i, e_j) &= 0 \end{aligned}$$

for $1 \le i \le n$ and $j \ne i \pm n$, and that

$$Q(x, y) = (x_1, \ldots, x_{2n}) \begin{pmatrix} O & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix}$$

Exercise 8.2. If we set $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{X \in \operatorname{Mat}_{2n}(\mathbb{C}) : X^t M + MX = 0\}.$$

We now describe

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in \operatorname{Mat}_n(\mathbb{C})$. Note that

$$X^t = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix}.$$

We have $X^t M + MX = 0$, so

$$\begin{pmatrix} -C^t & A^t \\ -D^t & B^t \end{pmatrix} + \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = 0.$$

Therefore, $B^t = B$, $C^t = C$, $A^t + D = 0$, and $D = -A^t$.

The Cartan subalgebras of $\mathfrak{sp}_{2n}(\mathbb{C})$ are

$$\begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix},$$

where the basis consists of $H_i = E_{ii} - E_{i+n,i+n}$ for $1 \leq i \leq n$. Let $\{L_j\} \subseteq \mathfrak{h}^*$ be a biorthogonal basis, so that $\langle L_j, H_i \rangle = \delta_{ij}$. Let

$$\begin{aligned} X_{ij} &= E_{ij} - E_{n+j,n+i} \\ y_{ij} &= E_{i,n+j} - E_{j,n+i} \\ Z_{ij} &= E_{n+i,j} + E_{n+j,i} \\ U_i &= E_{i,n+i} \\ V_i &= E_{n+i,i}. \end{aligned}$$

Exercise 8.3. Let $H \in \mathfrak{h}$. Show that

$$\begin{aligned} [H, X_{ij}] &= (L_i - L_j)(H) X_{ij} \\ [H, Y_{ij}] &= (L_i + L_j)(H) Y_{ij} \\ [H, Z_{ij}] &= (-L_i - L_j)(H) Z_{ij} \\ [H, U_i] &= 2L_i(H) U_i \\ [H, V_i] &= -2L_i(H) V_i. \end{aligned}$$

Show also that these are the only eigenvectors.

The roots for $\mathfrak{sp}_{2n}(\mathbb{C}$ are

$$\pm L_i \pm L_j \in \mathfrak{h}^*.$$

The root subspaces are

- span$\{X_{ij}, X_{ji}, H_i - H_j\}$
- span$\{Y_{ij}, Z_{ij}, H_i + H_j\}$
- span$\{U_i, V_i, H_i\}$.

The positive primitive roots are

$$R = \{L_i + L_j\}_{i \leq j} \cup \{L_i - L_j\}_{i<j}.$$

Remark. $\mathfrak{sp}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$.

We now describe the irreducible modules corresponding to the weights $L_1 + \cdots + L_k$. To this end, we construct $\varphi_k : \Lambda^k V \to \Lambda^{k-2} V$ by defining

$$\varphi_k(v_1, \ldots, v_k) = \sum_{i<j} Q(v_i, v_j) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k.$$

We note that φ_k is a contraction.

Theorem 8.4. *The kernel of φ_k is an irreducible module with $L_1 + \cdots + L_k$.*

2. Orthogonal Lie Algebras

Definition. Let V be an m-dimensional vector space over $\mathbb{C}$ with a nondegenerate, positive-definite, symmetric, sesquilinear form $Q : V \times V \to \mathbb{C}$. The *orthogonal Lie algebra* is defined to be the collection

$$\mathfrak{so}_m(\mathbb{C}) = \{A \in \operatorname{End} V : Q(Ax, y) + Q(x, Ay) = 0\}.$$

We now derive the matrix representation of $\mathfrak{so}_{2n}(\mathbb{C})$. Let $m = 2n$ and find a basis $e_1, \ldots, e_{2n}$ for V such that

$$\begin{aligned} Q(e_i, e_{i+n}) &= Q(e_{i+n}, e_i) = 1 \\ Q(e_i, e_j) &= 0 \text{ if } i \pm n \\ Q(x, y) &= (x_1, \ldots, x_{2n} \left(x_1, \ldots, x_2 n\right) \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix}. \end{aligned}$$

Setting $M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, we see that

$$\mathfrak{so}_{2n}(\mathbb{C}) = \{X \in \operatorname{Mat}_{2n}(\mathbb{C}) : X^t M + MX = 0\}.$$

Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that

$$\begin{pmatrix} C^t & A^t \\ D^t & B^t \end{pmatrix} + \begin{pmatrix} C & D \\ A & B \end{pmatrix} = 0.$$

It then follows that $C^t = -C$, $B^t = -B$, and $A^t + D = 0$.

We also derive the matrix representation of $\mathfrak{so}_{2n+1}(\mathbb{C})$. We let $m = 2n + 1$ and find a basis $e_1, \ldots, e_{2n+1}$ such that

$$\begin{aligned} Q(e_i, e_{i+n}) &= Q(e_{i+n}, e_i) \text{ for } 1 \leq in \\ Q(e_{2n+1}, e_{2n+1}) &= 1 \\ Q(e_i, e_j) &= 0 \text{ otherwise} \end{aligned}$$

$$Q(x, y) = (x_1, \ldots, x_{2n+1}) \begin{pmatrix} O & I_n & O \\ I_n & O & O \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{2n+1} \end{pmatrix}.$$

Setting $M = \begin{pmatrix} O & I_n & O \\ I_n & O & O \\ 0 & 0 & 1 \end{pmatrix}$, we see that

$$\mathfrak{so}_{2n+1}(\mathbb{C}) = \{X \in \operatorname{Mat}_{2n+1}(\mathbb{C}) : X^t M + MX = 0\}.$$

Exercise 8.5. Prove that if

$$X = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & J \end{pmatrix}$$

is in $\mathfrak{so}_{2n+1}(\mathbb{C})$, then $B^t = -B$, $C^t = -C$, $A^t + D = 0$, $E = -H$, $F = -G^t$, and $J = 0$.

The Cartan matrix of $\mathfrak{so}_{2n}(\mathbb{C})$ consists of diagonal matrices

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix}$$

with a basis consisting of $H_i = E_{ii} - E_{n+i,n+i}$ for $1 \leq i \leq n$. The Cartan matrix of $\mathfrak{so}_{2n+1}(\mathbb{C})$ consists of diagonal matrices

$$\begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_n & & & & \\ & & & -\lambda_1 & & & \\ & & & & \ddots & & \\ & & & & & -\lambda_n & \\ & & & & & & 0 \end{pmatrix}$$

with a basis consisting of $H_i = E_{ii} - E_{n+i,n+i}$ for $1 \leq i \leq n$.

Let $\{L_j\}$ be a biorthogonal basis in $\mathfrak{h}^*$. We shall first compute the eigenvectors for the action of $\mathfrak{h}$ in the $m = 2n$ case. We first note that

$$\begin{aligned} X_{ij} &= E_{ij} - E_{n+j,n+i} \\ Y_{ij} &= E_{i,n+j} - E_{j,n+i} \\ Z_{ij} &= E_{n+j,j} - E_{n+i,i}. \end{aligned}$$

We remark that that Y_{ij} and Z_{ij} are different from the symplectic case. Therefore

$$\begin{aligned} [H, X_{ij}] &= (L_i - L_j)(H) X_{ij} \\ [H, Y_{ij}] &= (L_i + L_j)(H) Y_{ij} \\ [H, Z_{ij}] &= (-L_i - L_j)(H) Z_{ij}. \end{aligned}$$

The roots are

$$\{\pm L_i \pm L_j\}_{i \neq j} \subseteq \mathfrak{h}^*.$$

The positive roots are

$$R_+ : \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j}.$$

In the case $m = 2n+1$, we take the same X_{ij}, Y_{ij}, and Z_{ij} and add

$$\begin{aligned} U_i &= E_{i,2n+1} - E_{2n+1,i} \\ V_i &= E_{n+i,2n+1} - E_{2n+1,n+i}. \end{aligned}$$

Then

$$\begin{aligned} [H, U_i] &= L_i(H) U_i \\ [H, V_i] &= -L_i(H) V_i. \end{aligned}$$

The roots are

$$\{\pm L_i \pm L_j\}_{i \neq j} \cup \{\pm L_i\}.$$

The positive roots are

$$R_+ : \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \{L_i\}_i.$$

3. Real Lie Algebras

Let L_0 be a real Lie algebra. Recall that the *complexification* of L_0 is $L = L_0 \otimes_{\mathbb{R}} \mathbb{C}$. We are interested in finding real Lie subalgebras of L whose complexification is isomorphic to L.

Example. Let $L = \mathfrak{sl}_n$. Then $L = \operatorname{span}_{\mathbb{C}}\{H, e, f\}$, and so we define

$$L_0 = \operatorname{span}_{\mathbb{R}}\{H, e, f\}.$$

We define a Lie algebra

$$\mathfrak{su}_2(\mathbb{R}) = \operatorname{span}_{\mathbb{R}} \left\{ A = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix} \right\}.$$

Exercise 8.6. *$H = -2i \cdot H$, $e = B - iC$, $f = -B - iC$.*

So, $\mathfrak{su}_2 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$. Then

$$L_0 = \operatorname{span}_{\mathbb{R}}\{H, e, f\}$$

is split form for $\mathfrak{sl}_2(\mathbb{C})$. Therefore, $\mathfrak{su}_2$ is a *compact form* because $\mathfrak{su}_2$ is a Lie algebra for compact Lie group.

4. Real Forms

Let L be a complex Lie algebra. We are looking for a Lie subalgebra L_0 over $\mathbb{R}$ such that $L_0 \otimes_{\mathbb{R}} \mathbb{C} = L$.

Example. $L = \mathfrak{sl}_2(\mathbb{C})$. $L_0 = \mathfrak{sl}_2(\mathbb{R})$. Another one:

$$\mathfrak{su}_2 = \operatorname{span} \left\{ \begin{pmatrix} i/2 & 0 \\ 0 & i/2 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix} \right\}.$$

Example. The generalization of the above example is as follows: $L = \mathfrak{sl}_n(\mathbb{C})$, $L_0 = \mathfrak{sl}_n(\mathbb{R})$, and

$$\mathfrak{su}_n = \{A \in \mathfrak{sl}_n(\mathbb{C}) : \overline{A^t} = -A\}.$$

There are other real forms of semisimple complex Lie algebras, but these two are called *extreme.*

Let L be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $L_0 \subseteq L$, and $\mathfrak{h}_0 \subseteq \mathfrak{h}$.

Definition. L is called a *complexification* of L_0, and L_0 a *real form* of L. L_0 is a *split* form if, for all $\alpha \in R \subseteq \mathfrak{h}^*$, the restriction $\alpha|_{\mathfrak{h}_0}$ is real. L_0 is a *compact* form if, for all $\alpha \in R \subseteq \mathfrak{h}^*$, the restriction $\alpha|_{\mathfrak{h}_0}$ is pure imaginary.

Proposition 8.7. *Let L be a complex semisimple Lie algebra. The following statements are equivalent.*

(i) For all $\alpha \in R \subseteq \mathfrak{h}^*$, the restriction $\alpha|_{\mathfrak{h}_0}$ is pure imaginary, and the subalgebra of L_0 generated by $(L_\alpha \oplus L_{-\alpha}) \cap L_0$ is $\mathfrak{su}_2$.
(ii) The Killing form restricted to L_0 is negative-definite.
(iii) The real Lie group G_0 with Lie algebra L_0 is compact.

CHAPTER 9

Further Results

1. Witt Algebras

We begin this chapter by a discussion of *Witt algebras* (E. Cartan 1909). For every f and g in $\mathbb{C}[t,t^{-1}]$, the ring of Laurent polynomials, we define the bracket

$$\{f,g\} = f'g - fg'.$$

This is a Lie bracket coming from the Poisson bracket on $f(t)\frac{\partial}{\partial t}$. Call this algebra W_1. The collection of elements $e_n = -t^{n+1}$ for all $n \in \mathbb{Z}$ is a basis of W_1. We see that

$$\begin{aligned}
[e_n, e_m] &= (n+1)t^n \cdot t^{m+1} - (m+1)t^m t^{n+1} \\
&= (n-m)t^{m+n+1} \\
&= (m-n)e_{n+m}.
\end{aligned}$$

In particular,

$$[e_0, e_m] = me_m.$$

We then have

$$W_1 = \operatorname{span}\{e_i\}_{i<0} \oplus \mathbb{C}e_0 \oplus \operatorname{span}\{e_i\}_{i>0},$$

which is "similar" to $\mathfrak{sl}_2(\mathbb{C})$.

An extension of the above algebra is given by adding the "center" c such that $[c, e_n] = 0$ for all $n \in \mathbb{Z}$. This is called the *Virasoro algebra*, with the following multiplication table:

$$\begin{aligned}
[e_n, e_m] &= (m-n)e_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c \\
[e_{-m}, e_m] &= 2e_m + \frac{m^3-m}{12}c.
\end{aligned}$$

We now consider modules generated by vector v with the following condition:

$$\begin{aligned}
e_0 v &= hv \\
cv &= cv, \text{ where the } c \text{ on the right is a constant} \\
e_i v &= 0, \text{ where } i > 0
\end{aligned}$$

The module is generated by

$$e_{-i_n}e_{-i_{n-1}}\cdots e_{-i_1}v,$$

where $i_1, \ldots, i_n > 0$. (cf. highest weight modules) (Natural questions to ask: When is the module irreducible? What are the characters?)

2. Current Lie Algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and X a smooth manifold. Then $\mathscr{C}^\infty(X, \mathfrak{g})$ has a Lie algebra structure with the bracket

$$[f, g](x) = [f(x), g(x)].$$

The notation is $\mathfrak{g}^X$. The most important case is $X = \mathbb{S}^1$. Then $\mathfrak{g}^{\mathbb{S}^1}$ is a current Lie algebra.

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. We choose generators

$$e_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{C})$. Define $\varepsilon_i(t) \in \mathfrak{g}^{\mathbb{S}^1}$ by setting

$$\begin{aligned} \varepsilon_{3k}(t) &= \exp(2\pi ikt)e_0 \\ \varepsilon_{3k-1}(t) &= \exp(2\pi ikt)e_{-1} \\ \varepsilon_{3k+1}(t) &= \exp(2\pi ikt)e_1. \end{aligned}$$

This constitutes a topological basis. We see that $[\varepsilon_i, \varepsilon_j] = \alpha_{ij}\varepsilon_{i+j}$, where

$$\alpha_{ij} = \begin{cases} -1 & \text{if } j - i \cong -1 \bmod 3; \\ 0 & \text{if } j - i \cong 0 \bmod 3; \\ 1 & \text{if } j - i \cong 1 \bmod 3. \end{cases}$$

3. Kac-Moody Lie algebras

We start with a Cartan matrix: $A = (a_{ij})$, where $1 \leq i, j \leq n$ with $a_{ij} \in \mathbb{Z}$,

$$a_{11} = a_{22} = \cdots = a_{nn} = 2,$$

and $a_{ij} \leq 0$.

Here is a technical condition: We say that A is *symmetrizable* if there exist $b_1, \ldots, b_n > 0$ such that $(b_i a_{ij})$ is symmetric. Notation: $bA = (b_i a_{ij})$.

We define Lie algebra $\mathfrak{g}^A/\mathbb{C}$ with generators $e_1, \ldots, e_n$, $h_1, \ldots, h_n$, $f_1, \ldots, f_n$ and relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij}h_i \\ [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ij}h_j \\ [h_i, f_j] &= -a_{ij}h_j \\ (\operatorname{ad} e_i)^{-a_{ij}+1} \cdot e_j &= 0 \\ (\operatorname{ad} f_i)^{-a_{ij}+1} \cdot f_j &= 0. \end{aligned}$$

Assume A is irreducible. By permuting rows and columns we cannot get a block-diagonal matrix.

Let us now consider the properties of $\mathfrak{g}^A$. We first define *multigrading* on $\mathfrak{g}^A$:

$$\begin{aligned} \deg h_i &= (0, \ldots, 0) \\ \deg e_i &= (0, \ldots, 1, 0, \ldots, 0) \\ \deg f_i &= (0, \ldots, -1, 0, \ldots, 0). \end{aligned}$$

$$\mathfrak{g}^A = \bigoplus \mathfrak{g}_{(i_1, \ldots, i_n)}.$$

Theorem 9.1. *The following holds:*

- *(i)* *If* $\operatorname{rank} A = n$, *then* $\mathfrak{g}^A$ *is simple. If* $\operatorname{rank} A = n-k$, $k > 0$, *then* $\mathfrak{g}^A$ *has* k*-dimensional center* $Z \subseteq \mathfrak{g}^A_{(0,\dots,0)}$ *and* $\mathfrak{g}^A/Z$ *does not have proper graded ideals.*
- *(ii)* $\mathfrak{g}^A = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, *where*
  - $\mathfrak{n}_-$ *is generated by* (f_i)
  - $\mathfrak{h}$ *is generated by* (h_i)
  - $\mathfrak{n}_+$ *is generated by* (e_i).

  Furthermore, $\mathfrak{g}^A_{(i_1,\dots,i_n)} = \{0\}$ *if* $(i_1,\dots,i_n)$ *contains both positive and negative indices.*
- *(iii)* $\dim \mathfrak{h} = n$.

□

We shall classify $\mathfrak{h}^A$ is as follows:

- (a) bA is positively defined
- (b) bA is nonnegatively defined of rank $n-1$.
- (c) rest.

Theorem 9.2 (Kac-Moody)**.** *The above classification scheme has the following concrete description*

- *(a) Simple finite-dimensional Lie algebra.*
- *(b)* $\dim Z = 1$. *and* $\mathfrak{g}^A/Z$ *is isomorphic to*
  - *(i)* $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$, *where* $\mathfrak{g}$ *is simple finite-dimensional Lie algebra, or*
  - *(ii) subalgebra of* $\oplus_{l=-\infty}^{\infty} \mathfrak{g}(l) \otimes t^l$, *where* $g(l)$ *is eigenspace of automorphism* $\theta : \mathfrak{g} \to \mathfrak{g}$ *of order 2 or 3.*
- *(c) If eigenvalues of* bA *are negative or the rank of* A *is at most* $n-2$, *then the dimension of*

$$\bigoplus_{i_1+\dots+i_n=k} \mathfrak{g}^A_{(i_1,\dots,i_n)}$$

grow exponentially depending on k.

4. Lie Groups and Lie Algebras

Definition. A *Lie group* is a topological group that is a smooth manifold where the group operation $(x,y) \mapsto xy$ and the inverse operation $x \mapsto x^{-1}$ are smooth.

Proposition 9.3. *Let* G *be connected. Then any neighborhood* U *of* e *generates* G.

Definition. A map $\rho : G \to H$ of Lie groups is a *homomorphism* if ρ is a group homomorphism and a smooth map.

Theorem 9.4. *Let* G *and* H *be Lie groups. If* G *is connected, then the homomorphism* $\rho : G \to H$ *is uniquely determined by the tangent map* $(d\rho)_e : T_eG \to T_eH$. □

For any $g \in G$, define $\psi_g : G \to G$ by $\psi_g(h) = ghg^{-1}$. We have the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\psi_g"'] & H \arrow[d, "\psi_{\rho(g)}"] \\ G \arrow[r, "\rho"] & H \end{tikzcd}$$

Setting $H = G$, we have

$$(d\psi_g)_e : T_eG \to T_eG,$$

for $\psi_g(e) = e$. We write

$$\mathrm{Ad}(g) = (d\psi_g)_e$$

for the *adjoint representation* $\mathrm{Ad} : G \to T_eG$. Note that

$$\mathrm{Ad}(g_1g_2) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2).$$

We also have the following commutative diagram:

$$\begin{tikzcd}
T_eG \arrow[r, "(d\rho)_e"] \arrow[d, "\mathrm{Ad}(g)"'] & T_e \arrow[d, "\mathrm{Ad}(\rho(g))"] \\
Te_G \arrow[r, "(d\rho)_e"] & T_eH.
\end{tikzcd}$$

The tangent map to Ad is $\mathrm{ad} : T_eG \to \mathrm{End}\, T_eG$, which satisfies the commutative diagram

$$\begin{tikzcd}
T_eG \arrow[r, "(d\rho)_e"] \arrow[d, "\mathrm{ad}(v)"'] & T_eH \arrow[d, "\mathrm{ad}(d\rho(v))"'] \\
T_eG \arrow[r, "(d\rho)_e"] & T_eH.
\end{tikzcd}$$

We define a Lie bracket on T_eG by

$$[X, Y] = \mathrm{ad}(x)(Y)$$

for each $X, Y \in T_eG$. Then

$$d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)].$$

Example. Let $G = GL_n(\mathbb{R})$, so that $T_eG = \mathrm{Mat}_n(\mathbb{R})$. Let X and Y be tangent vectors at e. There exists a smooth curve $\gamma : I \to G$ on an open interval I containing 0 such that $\gamma(0) = e$ and $\gamma'(0) = X$. We observe that

$$\begin{aligned}
[X, Y] &= \mathrm{ad}(X)(Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\gamma(t))(Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} \gamma(t)Y\gamma(t)^{-1} \\
&= \gamma'(0)Y\gamma(t)^{-1} + \gamma(t)Y(\gamma(t)^{-1})'_{t=0}.
\end{aligned}$$

Let $M(t)$ be an m-by-m matrix depending on t. Then

$$(M^{-1}(t))' = -M(t)^{-1}M(t)'M(t)^{-1}.$$

Why is that? Since $M(t)M(t)^{-1} = I$, we differentiate each side to get

$$M(t)'M(t)^{-1} + M(t)(M(t)^{-1})' = 0.$$

We then have

$$\begin{aligned}
[X, Y] &= \gamma'(0)Y\gamma(0) - \gamma(0)Y\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1} \\
&= XY - YX.
\end{aligned}$$

Exercise 9.5. Check $[Y, X] = -[X, Y]$ and Jacobi identity.

Let M be a connected manifold. Recall that M is *simply connected* if $\pi_1(M) = \{e\}$.

Theorem 9.6. *Let G and H be Lie groups. If G is connected and simply connected, then every linear map $T_eG \to T_eH$ is a tangent map to a homomorphism $G \to H$ if and only if it preserves the Lie bracket.* □

So, we went from Lie groups to Lie algebras. Let us now go from Lie algebras to Lie groups. Before we do so, we write

$$\mathfrak{g} = T_eG$$

to denote the Lie algebra corresponding to a Lie group G.

Let $X \in \mathfrak{g}$. We define $m_g : G \to G$ by

$$m_g(h) = gh.$$

A *vector field* on G is defined by

$$v_x(g) = (m_g)_*(X).$$

We integrate the vector field. Let p be a point on a manifold M. There exists a unique map $\varphi : I \to M$ such that $\varphi(0) = p$ and $\varphi'(t) = v(\varphi(t))$. If we do this for $M = G$ and v_X, we get $\varphi_X : \mathbb{R} \to G$ such that

(1) $\varphi_X(0) = e$
(2) φ_X is a homomorphism of Lie groups
(3) $\varphi'_X(0) = X$.

Definition. $\varphi_X(\mathbb{R})$ is a *one-parameter subgroup* of G.

So, each vector $X \in T_eG$ is tangent to one-parameter group.

Definition. The *exponential map* $\exp : \mathfrak{g} \to G$ is defined by

$$\exp(X) = \varphi_X(1).$$

Theorem 9.7. *The exponential map is the unique map from $\mathfrak{g}$ to G such that*

(1) $0 \mapsto e$
(2) $(\exp_*)_0 : T_eG \to T_eG$ *is the identity map.*
(3) The restrictions of exp *on straight lines passing through 0 are one-parameter subgroups.*

Therefore, we may cover each neighborhood of e by $\exp(\mathfrak{g})$. □

Given bracket $[\cdot,\cdot]$ on $\mathfrak{g}$, can we construct the group operation in G? To this end, we let $G = GL_n(\mathbb{R})$ and $X \in \mathrm{Mat}_n(\mathbb{R})$. Then

$$\exp X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

Take $\exp X \cdot \exp Y$. When does it belong to $\exp(\mathfrak{g})$? This is equivalent to solving

$$\exp X \cdot \exp Y = \exp Z$$

for Z. By noncommutativity, the identity

$$\exp X \cdot \exp Y = \exp(X + Y)$$

does not always hold.

The answer is given by the *Campbell-Hausdorff formula*, which roughly states that Z can be expressed in terms of X, Y, and $[\cdot,\cdot]$. Let $g \in Mat_n(\mathbb{R})$. We have

$$\log(g) = gI - \frac{1}{2}(g-I)^2 + \frac{1}{3}(g-I)^3 - \cdots ,$$

which converges if $g \sim 0$. Set

$$Z = \log(\exp(X)\exp(Y)).$$

note that

$$\begin{aligned} & \exp(X)\exp(Y) \\ = \; & \left(1 + X + \frac{1}{2!}X^2 + \cdots\right)\left(1 + Y + \frac{1}{2!}Y^2 + \cdots\right) \\ = \; & \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots\right) \end{aligned}$$

There are no simple descriptions of the terms of Z. All available ones are rather complicated combinatorial descriptions.[1]

Proposition 9.8. *Let G be a Lie group and $\mathfrak{g}_1$ a Lie subalgebra of $\mathfrak{g} = T_eG$. Then $\exp_{\mathfrak{g}_1} = G_1$ is a Lie subgroup of G.* □

If G_1 and G_2 are Lie groups and G_1 simply connected, then the linear map $\mathfrak{g}_1 \mapsto \mathfrak{g}_2$ is a Lie algebra homomorphism if and only if the corresponding map $G_1 \to G_2$ is a homomorphism.

Which Lie groups are simply connected?

Proposition 9.9. *For $n \geq 1$, $SL_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ are connected and simply connected. Also, $SO_n(\mathbb{C})$ is connected but $\pi_1(SO_2(\mathbb{C})) = \mathbb{Z}$ and $\pi_1(SO_n(\mathbb{C})) = \mathbb{Z}/2$ for $n \geq 3$.* □

Exercise 9.10. $SO_2(\mathbb{C}) \cong \mathbb{C}^*$.

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[1] Some of them are available in Reutenaeur, *Free Lie Algebras*.