Po Lam Yung: A "random theorem" in harmonic analysis

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(This was my first time live texing, so I apologize for the type-o's and rough formatting.)

1 Intro

Theorem 1. *(Marcinkiewicz-Zygmund) Suppose $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bdd linear for some $p \in [1, \infty]$. Then we have*

$$||(\sum_{j=1}^{\infty} |Tf_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^{\infty} |f_j|^2)^{1/2}||_{L^p}$$

for any sequence f_j. Here $M = ||T||$.

(Recall from Harmonic Analysis, that if you replace T by maximal function, this inequality holds for $p \in (1, \infty)$.)

This can be used in the proof of the following theorem:

Theorem 2. *(Fefferman ball multiplier) Suppose 1_B is the characteristic function of the unit ball in $\mathbb{R}^n$. Define a map $S : L^2 \to L^2$ by $\hat{S}f(\xi) = 1_B(\xi)\hat{f}(\xi)$. Then S does not extend to a bounded linear map $L^p \to L^p$ if $p \neq 2$ and $n > 1$.*

For comparison,

$$\hat{S_0}f(\xi) = 1_Q(\xi)\hat{f}(\xi)$$

then $S_0 : L^p \to L^p \ \forall p \in (1, \infty)$. So balls are bad, but cubes are good (can decompose into intervals).

2 Proof of Fefferman's Result

Notation:

- $B(x, R)$ = ball centered at $u \in \mathbb{R}^n$ of radius R
- $B_R = B(0, R)$
- $B = B_1$
- If $u \in \mathbb{R}^n$ is a unit vector, $Hu = \{\xi \in \mathbb{R}^n : \xi \cdot u > 0\}$ is the half space into which u points.
- $\hat{S_A}f(\xi) = 1_A(\xi)\hat{f}(\xi)$ for any $A \subseteq \mathbb{R}^n$ (so that the S in the theorem is just S_B.

Proof. Step 1: Suppose $S = S_B : L^p \to L^p$ for some $p \neq 2$, $n > 1$. WLOG, assume $p < 2$ (if it is bounded in this case, then it is bounded for $p > 2$ by duality, since we have a self-adjoint operator) and $n = 2$.

Then, by Marcinkiewicz-Zygmund, we have

$$||(\sum_{j=1}^{\infty} |S_B f_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^{\infty} |f_j|^2)^{1/2}||_{L^p}.$$

Step 2: This implies

$$||(\sum_{j=1}^{\infty} |S_{B_R} f_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^{\infty} |f_j|^2)^{1/2}||_{L^p}.$$

(Because for each $R > 0$, define $\delta_R f(x) = f(Rx)$ (dilation by R). Then $S_B = \delta_R^{-1} \circ S_{B_R} \circ \delta_R$.)

Now, M is independent of $R > 0$.

Step 3: Suppose we have a sequence of unit vectors $u_j \in \mathbb{R}^n$.

CLAIM:

$$||(\sum_{j=1}^{\infty} |S_{B(Ru_j,R)} f_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^{\infty} |f_j|^2)^{1/2}||_{L^p}$$

Picture: $B(Ru_j, R)$ is the ball centered at Ru_j that just brushes the boundary of the half plane Hu_j.

This follows because $S_{B_R} f(x) = e^{2\pi i R u_j \cdot x} S_{B_r}(x^{-2\pi i R u_j \cdot x} f(x))$ (can check by taking Fourier transform of both sides), and the exponentials disappear when you take absolute values.

Step 4: Let $R \to \infty$. Then the $B(Ru_j, R)$ increase to fill the half space Hu_j. Hence,

$$||(\sum_{j=1}^{\infty} |S_{Hu_j} f_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^{\infty} |f_j|^2)^{1/2}||_{L^p}$$

for all choices of unit vectors $u_j \in \mathbb{R}^n$ and all choices of measurable functions f_j. So you have so many things to pick! To find a contradiction, yo just want to make a clever choice so that the inequality fails.

Pause in proof

□

3 Besicovich Sets

For all $\epsilon > 0$, there exists $E \subseteq \mathbb{R}^2$ mess such that $|E| < \epsilon$ and there exists rectangles $R_1, \ldots, R_N$ of sizes $1 \times \frac{1}{N}$ such that $E = \cup_{j=1}^N R_j$. (Think: the rectangles must overlap a lot so the measure of E is small.) Furthermore, there exists a translate $\hat{R}_j$ of R_j along the longer side of R_j by distance 2 such that the R_j are pairwise disjoint. (So while the R_j's overlap a lot, the $\hat{R}_j$'s are disjoint.)

This gives a bound on the Hilbert transform: For each unit vector $u \in \mathbb{R}^2$, take a rectangle R_u of dimensions $1 \times a$ such that the side of length 1 is parallel to u. Then $|S_{Hu}(1_{Ru})(x)| \geq C 1_{\hat{Ru}}(x)$ where $\hat{Ru}$ is Ru translated by 2 in the opposite direction from u.

4 Finish Proof of Fefferman

Proof. Given any $\epsilon > 0$, let $E = \cup_{j=1}^N R_j$. Let u_j be the direction of the longer side of R_j (pick either direction). Let $f_j = 1_{R_j}$ if $j \leq N$ or 0 if $j > N$. By Step 4, we know

$$||(\sum_{j=1}^N |S_{Hu_j} f_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^N |f_j|^2)^{1/2}||_{L^p}$$

But this inequality cannot possibly hold! The left hand side of this inequality is bounded below by

$$||(\sum_{j=1}^N |1_{\hat{R}_j}|^2)^{1/2}||_{L^p} = ||(\sum_{j=1}^N |1_{\hat{R}_j}|)^{1/2}||_{L^p} = (\int (\sum_{j=1}^N 1_{\hat{R}_j})^{p/2} dx)^{1/p} = 1^{1/p} = 1$$

(Since the $\hat{R}_j$'s are disjoint rectangles.)

On the other hand, since $p < 2$, $p/2 < 1$, so the RHS is...

$$(\int ((\sum_{j=1}^N |f_j|^2)^{p/2} dx)^{1/p} = (\int_E ((\sum_{j=1}^N |f_j|^2)^{p/2} dx))^{1/p}$$

by Holder's inequality, $\frac{2}{p} + \frac{1}{q} = 1$,

$$\leq ((\int_E (\sum_{j=1}^N |f_j|^2) dx)^{2/p} (\int_E 1^q)^{1/q})^{1/p} = |E|^{1/qp} < \epsilon \to 0$$

□

5 Proof of Marcinkiewicz-Zygmund

Theorem 3. *(Marcinkiewicz-Zygmund) Suppose $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bdd linear for some $p \in [1, \infty]$. Then we have*

$$||(\sum_{j=1}^\infty |Tf_j|^2)^{1/2}||_{L^p} \leq M||(\sum_{j=1}^\infty |f_j|^2)^{1/2}||_{L^p}$$

for any sequence f_j. Here $M = ||T||$.

There are two proofs.

Proof. By Monotone convergence theorem, it's enough to prove for finite sums.

Unitary trick (for linear operators): For each $\omega = (\omega_1, \dots, \omega_N) \in \mathbb{C}^N$, consider

$$f_\omega(x) = \sum_{j=1}^{n} \bar{\omega_j} f_j(x)$$

Because T is a bounded operator,

$$||Tf_\omega||_p^p \leq M^p ||f_\omega||_p^p$$

Now, here is the crucial (seemingly innocuous) step:

$$Tf_\omega(x) = T(\sum_{j=1}^{n} \bar{\omega_j} f_j(x)) = \sum_{j=1}^{n} \bar{\omega_j} Tf_j(x).$$

Therefore,

$$\int |\sum_{j=1}^{n} \bar{\omega_j} Tf_j(x)^p dx \leq M^p \int |\sum j = 1^N \bar{\omega_j} f_j(x)|^p dx$$

Now, integrate over $\omega \in \mathbb{S}^{2N-1} \subseteq \mathbb{C}^N$. Since everything is nonnegative, we can interchange order of integration:

$$\int_{\mathbb{R}^n} \int_{|\omega|=1} |\sum_{j=1}^{n} \bar{\omega_j} Tf_j(x)^p d\omega dx \leq M^p \int_{\mathbb{R}^n} \int_{|\omega|=1} |\sum j = 1^N \bar{\omega_j} f_j(x)|^p d\omega dx$$

Lemma 5.1. *For each $a = (a_1, \dots, a_n) \in \mathbb{C}^N$,*

$$\int_{|\omega|=1} |sum_{j=1}^N \bar{\omega_j} a_j|^p d\omega = (\sum j = 1^N |a_j|^2)^{1/2}$$

This follows because the $d\omega$ measure is rotation invariant. So we can rotate a so that $a = (|a|, 0, \dots, 0)$.

Back to the proof... we get

$$\int_{\mathbb{R}^n} ((\sum j = 1^N |Tf_j(x)|^2)^{1/2})^p \leq M^p \int_{\mathbb{R}^n} ((\sum_{j=1}^{n} |f_j(x)|^2)^{1/2})^p dx$$

□

There is another proof that uses more probability theory and randomness.