Challenge Problem Set 1, Math 292 Spring 2012

Eric A. Carlen[1]
Rutgers University

January 31, 2012

1 Introduction

This challenge problem set concerns the behavior of equations

$$x'(t) = v(x(t)) \tag{1.1}$$

where we do not have the Lipschitz condition

$$|v(x) - v(y)| \leq L \ .$$

For example, consider the case

$$v(x) = x \ln x \tag{1.2}$$

on the interval $(0, 1)$.

1.(a) We know that for any $x \in (0, 1)$, the formula

$$t(x) = \int_{x_0}^{x} \frac{1}{s \ln s} \mathrm{d}s$$

gives us, after inverting to find $x(t)$, one solution of $x' = x \ln x$ with $x(0) = x_0$. Do the integral in this case to find $t(x)$. Show that

$$\lim_{x \to 0} t(x) = \infty \ ,$$

so that the solution keeps moving closer to 0 ($v(x)$ is negative), but never reaches it. Find an explicit formula for the solution.

(b) Show that the function $x(t)$ found in part **(a)** is the only solution of $x' = x \ln x$ with $x(0) = x_0$. The discussion on page 11 of the text applies in this case.

2. (a) Now consider

$$v(x) = -x \ln x \tag{1.3}$$

on the interval $(0, 1)$. Because we have included a minus sign this time, $v(x)$ is positive on $(0, 1)$, and solutions of (1.1) for this choice of $v(x)$ are monotone increasing. For each $x_0 \in (0, 1)$, there is exactly one solution of $x' = -x \ln x$ with $x(0) = x_0$. Find an explicit formula for it, and prove that this solution is unique.

(d) Things are more delicate for $x' = -x \ln x$ with $x(0) = 0$. Clearly, one solution is $x(t) = 0$ for all $t \geq 0$. Is this the only solution, or is the fact that $v(x)$ is not differentiable at $x = 0$ a sign of real trouble, as iti s with $v(x) = x^{2/3}$?

In fact, for this choice of $v(x)$, the constant solution $x(t) = 0$ for all $t \geq 0$ *is* the only solution of $x' = -x \ln x$ with $x(0) = 0$. Prove this is the case, by contradiction, supposing that there exists some solution $y(t)$ with $y' = -y \ln y$ and $y(0) = 0$, but $y(t_0) \neq 0$ for some t. Since $y(t)$ is continuous, we may assume $y(t_0) \in (0, 1)$. Show that the function

$$z(t) = y(a - t)$$

is a solution of

$$z' = z \ln z \qquad \text{with} \qquad z(0) = y(a) \in (0, 1) \ .$$

Conclude from the results in Problem **1** that $z(t) > 0$ for all t, and that while $\lim_{t\to\infty} z(t) = 0$, it takes $z(t)$ an infinite time to reach 0. On the other hand, $y(0) = z(a) = 0$. Thus the existence of a solution $y(t)$ with $y(t0)) > 0$ for any t_0 leads to a contradiction.

3. Prove the following theorem:

1.1 THEOREM. *Let $v(x)$ be strictly positive and differentiable on an interval (a, b) with $v(a) = v(b) = 0$. Then the only solution of $x'(t) = v(x(t))$, $t > 0$, with $x(0) = 0$ is $x(t) = a$ for all t if and only if for some (and hence all) $x_) \in (a, b)$,*

$$\int_0^{x_0} \frac{1}{v(s)} \mathrm{d}s = \infty \ .$$

From the point of view of this theorem, the uniqueness problem for $x' = x^{2/3}x$, $x(0) = 0$, is due to the fact that $\int_0^{x_0} s^{-2/3} \mathrm{d}z$ is a convergent improper integral for any $x_0 > 0$.

4. Let $0 < x_0 < y_0 < 1$. Let $x(t)$ be the unique solution of $x' = lnx$ with $x(0) = x_0$, and let $y(t)$ be the unique solution of $y' = ylny$ with $y(0) = y_0$. Show that $|y(t) - x(t)|$ goes to zero faster than any exponential. That is, that is, there is no $L > 0$ for which

$$|y(t) - x(t)| \geq |y_0 - x_0| e^{-tL} \ .$$

Extra Credit: Consider any continuous function $f(t)$ defined on $(0, \infty)$ with $f(0) = 1$ and with f strictly monotone decreasing to 0; i.e., $\lim_{t\to\infty} f(t) = 0$ and for $s < t$, $f(s) > f(t)$. Show that no matter how fast $f(t)$ converges to zero, given $x_0 < y_0 \in (0, 1)$, there is a function $v(x)$, strictly positive and continuously differentiable on $(0, 1)$ so that with $x'(t) = v(x(t))$ with $x(0) = x_0$ and $y'(t) = v(y(t))$ with $y(0) = y_0$, for all sufficiently large t, one has

$$|y(t) - x(t)| < f(t) \ .$$

That is without the Lipschitz condition, solutions can "draw together" arbitrarily fast, so information about the initial condition gets wiped out arbitrarily fast.