SINGULAR INTEGRALS: INTRODUCTION AND L^2 THEORY

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This is some notes for an expository lecture given at the student analysis seminar at Rutgers, where an introduction to singular integrals was given, and some aspects of L^2 theory were discussed. Given the expository nature of the talk, we minimize the use of distribution theory wherever possible.

1. Fourier transform

The Fourier transform of an $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} d\xi.$$

Then we have

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

Also, if $f \in L^1 \cap L^2$, then $\widehat{f} \in L^2$, with

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}.$$

Hence one can extend the Fourier transform to an isometry on $L^2(\mathbb{R}^n)$. (This is sometimes called Phancherel's theorem.)

The Fourier transform interacts nicely with derivatives: for instance, if $f \in C_c^\infty$, then

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = 2\pi i \xi_j \widehat{f}(\xi).$$

In particular,

$$\widehat{\Delta f}(\xi) = -4\pi^2 |\xi|^2 \widehat{f}(\xi).$$

2. Convolution

The convolution of two functions $f, K \in C_c(\mathbb{R}^n)$ is defined by

$$f * K(x) = \int_{\mathbb{R}^n} f(x-y) K(y) dy.$$

It is clear that $f*K = K*f$, and one can show that the convolution extends to a bilinear map from $L^p \times L^q \to L^r$, if $1 \leq p, q, r \leq \infty$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

For example, if $f \in L^p$ and $K \in L^1$, where $1 \leq p \leq \infty$, then

$$\|f * K\|_{L^p} \leq \|K\|_{L^1} \|f\|_{L^p}. \tag{1}$$

The convolution interacts nicely with the Fourier transform:

$$\widehat{f * K}(\xi) = \widehat{f}(\xi)\widehat{K}(\xi),$$

which holds for example when f is a Schwartz function and K is a tempered distribution.

One question we will ask is whether the mapping $f \mapsto f * K$ is bounded on L^p, if K is a tempered distribution that is 'almost' in L^1.

3. Singular Integrals

We begin by some specific examples.

Example 1. *The Hilbert transform is the operator*

$$Hf(x) = P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} f(x-y) \frac{dy}{y},$$

defined initially for $f \in C_c^\infty(\mathbb{R})$.

This integral converges for all x if $f \in C_c^\infty(\mathbb{R})$, since

$$\begin{aligned}
\left| \int_{\varepsilon_1 \leq |y| \leq \varepsilon_2} f(x-y) \frac{dy}{y} \right| &= \left| \int_{\varepsilon_1 \leq |y| \leq \varepsilon_2} (f(x-y) - f(x)) \frac{dy}{y} \right| \\
&\leq \int_{\varepsilon_1 \leq |y| \leq \varepsilon_2} \frac{|f(x-y) - f(x)|}{|y|} dy \\
&\leq C(\varepsilon_2 - \varepsilon_1) \to 0 \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0.
\end{aligned}$$

Similarly, one can define the Riesz transforms on $\mathbb{R}^n$:

Example 2. *The Riesz transforms are defined by*

$$R_j f(x) = P.V. \int_{-\infty}^{\infty} f(x-y) \frac{y_j}{|y|^{n+1}} dy, \quad j = 1, \ldots, n,$$

defined initially for $f \in C_c^\infty(\mathbb{R}^n)$.

These are convolution operators; in other words, they are operators of the form $f \mapsto f * K$, where $K(y) = P.V. \frac{1}{\pi y}$ in the first case, and $K(y) = P.V. \frac{y_j}{|y|^{n+1}}$ in the second case. But these kernels barely fail to be in L^1. Nevertheless, one can prove:

Theorem 1. *These operators extend to bounded operators on L^p for all $1 < p < \infty$.*

We will discuss the proof in a moment. We remark, on the other hand, that these operators are not bounded on L^1 and L^∞. (c.f. (1).)

We take this opportunity to point out some applications of this theorem to the study of elliptic partial differential equations. We have, for instance, the following a priori estimate for the Laplace equation:

Theorem 2. *Suppose $u \in C_c^\infty$ on $\mathbb{R}^n$, and $\Delta u = f$. Then*

$$\|\nabla^2 u\|_{L^p} \leq C_p \|f\|_{L^p}$$

for all $1 < p < \infty$.

This is because one has

$$\widehat{R_j f}(\xi) = c_n \frac{\xi_j}{|\xi|} \widehat{f}(\xi) \quad \text{for } j = 1, \dots, n,$$

for all $f \in C_c^\infty$. In fact, one can extend R_j to $L^2(\mathbb{R}^n)$ such that this continues to hold for all $f \in L^2(\mathbb{R}^n)$. Noe if u is as in the theorem, then

$$\widehat{\frac{\partial^2 u}{\partial x_j \partial x_k}} = -4\pi^2 \xi_j \xi_k \widehat{u} = -4\pi^2 \frac{\xi_j \xi_k}{|\xi|^2} |\xi|^2 \widehat{u} = -4\pi^2 \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f},$$

from which it follows that

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = c R_j R_k f.$$

Our desired assertion then follows from the boundedness of the Riesz transforms on L^p, $1 < p < \infty$.

The proof of Theorem 1 is usually in two parts. First one proves the theorem when $p = 2$. Then one proves the theorem for all other $1 < p < \infty$. In fact the second step works in a very general setting. We have:

Theorem 3. *Suppose that T is a bounded linear operator on $L^2(\mathbb{R}^n)$, and that there is a function $K(x, y)$, defined for all $x, y \in \mathbb{R}^n$ with $x \neq y$, such* *that for all $f \in L^2(\mathbb{R}^n)$ with compact support, one has*[1]

$$Tf(x) = \int_{\mathbb{R}^n} f(y)K(x,y)dy \quad \textit{for a.e. } x \notin \textit{supp}(f). \tag{2}$$

Suppose in addition that there exists a constant A such that for all $y, y_0 \in \mathbb{R}^n$, we have

$$\int_{|x-y|\geq 2|y-y_0|} |K(x,y) - K(x,y_0)|dx \leq A \tag{3}$$

and

$$\int_{|x-y|\geq 2|y-y_0|} |K(y,x) - K(y_0,x)|dx \leq A. \tag{4}$$

Then T extends[2] *to a bounded linear map on $L^p(\mathbb{R}^n)$, for all $1 < p < \infty$.*

In particular, we have the following L^p theorem for singular integral operators that are given by convolutions:

Corollary 1. *Suppose that T is a bounded linear operator on $L^2(\mathbb{R}^n)$, and that there is a function $K_0(y)$, defined for all $y \in \mathbb{R}^n$ with $y \neq 0$, such that for all $f \in L^2(\mathbb{R}^n)$ with compact support, one has*[3]

$$Tf(x) = \int_{\mathbb{R}^n} f(x-y)K_0(y)dy \quad \textit{for a.e. } x \notin \textit{supp}(f). \tag{5}$$

Suppose in addition that there exists a constant A such that for all $y \in \mathbb{R}^n \setminus \{0\}$, we have

$$|\nabla K_0(y)| \leq \frac{A}{|y|^{n+1}}.$$

Then T extends[4] *to a bounded linear map on $L^p(\mathbb{R}^n)$, for all $1 < p < \infty$.*

In fact, given such an operator T with kernel K_0, if one lets $K(x,y) = K_0(x-y)$, then the assumptions (2), (3) and (4) in Theorem 3 holds, and thus Theorem 3 applies.

We will not discuss the proof of Theorem 3 today. Rather, we turn to the L^2 theory of singular integrals: we will prove that the Hilbert transform is bounded on $L^2(\mathbb{R})$, and that the Riesz transforms are bounded on $L^2(\mathbb{R}^n)$. Once that is done, it is easy to see that if T were the Hilbert transform or the Riesz transforms, then condition (5) in Corollary 1 holds, with $K_0(y) =$

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[1] More precisely, this means that for almost every x not in the support of f, the integral converges absolutely and is equal to $Tf(x)$.

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$\frac{1}{\pi y}$ and $K_0(y) = \frac{y_j}{|y|^{n+1}}$ respectively. Thus Corollary 1 applies, and one concludes the proof of Theorem 1.

4. L^2 theory: the Fourier transform

The key to the L^2 theory of the Hilbert and Riesz transforms is the following:

$$\widehat{Hf}(\xi) = -i\text{sgn}(\xi)\widehat{f}(\xi) \quad \text{for all } f \in C_c^\infty(\mathbb{R}), \tag{6}$$

and

$$\widehat{R_j f}(\xi) = c_n \frac{\xi_j}{|\xi|}\widehat{f}(\xi) \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n). \tag{7}$$

In fact we have used the formula (7) in the proof of Theorem 2 already. These formula follow once we see that the Fourier transforms of the tempered distributions $P.V.\frac{1}{\pi y}$ and $P.V.\frac{y_j}{|y|^{n+1}}$ are $-i\text{sgn}(\xi)$ and $c_n\frac{\xi_j}{|\xi|}$ respectively. There are different ways of deriving these formula; we take them for granted for the moment, and we will come back and derive (6) later.

Now to prove for instance that the Hilbert transform extends to a bounded operator on $L^2(\mathbb{R})$, we just need to use Phancherel's theorem: for all $f \in C_c^\infty(\mathbb{R})$, we have

$$\|Hf\|_{L^2} = \|\widehat{Hf}\|_{L^2} = \| - i\text{sgn}(\xi)\widehat{f}\|_{L^2} = \|\widehat{f}\|_{L^2} = \|f\|_{L^2}.$$

Thus H does not only extend to be a bounded operator on $L^2(\mathbb{R})$; in fact it extends to be an isometry of $L^2(\mathbb{R})$. Similarly one can prove that the Riesz transforms extend to bounded operators on $L^2(\mathbb{R}^n)$.

5. Detour: Cauchy integrals and the Hilbert transform

We now turn to a proof of equation (6). We will not take the most direct route possible; rather, we derive it by relating the Hilbert transform to the Cauchy integral on the upper half plane.

First, we think of $\mathbb{R}$ as the real axis in the complex plane, and let U be the upper half space: $U = \{z \in \mathbb{C} \colon \text{Im } z > 0\}$. Given $f \in L^2(\mathbb{R})$, we define its Cauchy integral on the upper half plane by

$$Cf(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}dt, \quad z \in U.$$

This integral converges for each $z \in U$ since given any $z \in U$, the function $t \mapsto \frac{1}{t-z}$ is L^2 in t. It defines a holomorphic function on U. Equivalently, we have

$$Cf(z) = \int_0^\infty \widehat{f}(\xi) e^{2\pi i z\xi} d\xi, \quad z \in U,$$

since

$$\int_0^\infty e^{2\pi i(z-t)\xi} d\xi = \frac{1}{2\pi i(t-z)}. \tag{8}$$

(The integral $\int_0^\infty \widehat{f}(\xi)e^{2\pi i z\xi}d\xi$ converges since $\widehat{f}(\xi) \in L^2$ and $e^{2\pi i z\xi}$ is rapidly decreasing in ξ if $z \in U$.) Now for $y > 0$, let $C_y f(x) = Cf(x+iy)$. Then we have

$$\widehat{C_y f}(\xi) = \chi_{\{\xi>0\}} e^{-2\pi y\xi} \widehat{f}(\xi).$$

Hence if C_0 is the operator on $L^2(\mathbb{R})$ defined by

$$\widehat{C_0 f}(\xi) = \chi_{\{\xi>0\}} \widehat{f}(\xi),$$

then for all $f \in L^2(\mathbb{R})$, we have $C_y f \to C_0 f$ in $L^2(\mathbb{R})$ as $y \to 0$.

Now the kernel of C can be written as the sum of its real and imaginary parts:

$$\frac{1}{2\pi i(t-z)} = \frac{1}{2}\left(\frac{y}{\pi((x-t)^2+y^2)} + i\frac{x-t}{\pi((x-t)^2+y^2)}\right), \quad z = x+iy. \tag{9}$$

So if one defines, for $y > 0$,

$$P_y(x) = \frac{y}{\pi(x^2+y^2)} \quad \text{and} \quad Q_y(x) = \frac{x}{\pi(x^2+y^2)}, \tag{10}$$

then

$$C_y f(x) = \frac{1}{2}\left(f * P_y(x) + i f * Q_y(x)\right), \quad y > 0.$$

(The two convolutions converge since P_y and Q_y are L^2 functions for each $y > 0$.) The above convergence of $C_y f$ to $C_0 f$ can then be understood as follows. Taking the real and imaginary parts of (8), with $t = 0$, and using (9) and (10), we have

$$P_y(x) = \int_{-\infty}^\infty e^{-2\pi|y|\xi} e^{2\pi i x\xi} d\xi \quad \text{and} \quad Q_y(x) = \int_{-\infty}^\infty -i\text{sgn}(\xi) e^{-2\pi|y|\xi} e^{2\pi i x\xi} d\xi.$$

It follows that if $f \in L^2(\mathbb{R})$, then

$$f * P_y(x) = \int_{-\infty}^\infty \widehat{f}(\xi) e^{-2\pi|y|\xi} e^{2\pi i x\xi} d\xi$$

and

$$f * Q_y(x) = \int_{-\infty}^\infty -i\text{sgn}(\xi)\widehat{f}(\xi) e^{-2\pi|y|\xi} e^{2\pi i x\xi} d\xi.$$

(These integrals converge since $\widehat{f} \in L^2$.) As a result,

$$\widehat{f * P_y}(\xi) = e^{-2\pi|y|\xi}\widehat{f}(\xi) \quad \text{and} \quad \widehat{f * Q_y}(\xi) = -i\text{sgn}(\xi) e^{-2\pi|y|\xi}\widehat{f}(\xi).$$

If now Q_0 is the operator on $L^2(\mathbb{R})$ defined by

$$\widehat{Q_0 f}(\xi) = -i\text{sgn}(\xi)\widehat{f}(\xi),$$

then for all $f \in L^2(\mathbb{R})$, we have

$$f * P_y \to f \quad \text{and} \quad f * Q_y \to Q_0 f \quad \text{in } L^2(\mathbb{R})$$

as $y \to 0$. In particular

$$C_0 f = \frac{1}{2}(f + iQ_0 f)$$

for all $f \in L^2(\mathbb{R})$.

Now (6) is equivalent to the assertion that

$$Hf(x) = Q_0 f(x) \quad \text{for a.e. } x$$

for all $f \in C_c^\infty(\mathbb{R})$. This will follow if one can prove for every $f \in C_c^\infty(\mathbb{R})$, we have

$$H_\varepsilon f - f * Q_\varepsilon \to 0$$

in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, where

$$H_\varepsilon f(x) := \frac{1}{\pi} \int_{|y| \geq \varepsilon} f(x-y) \frac{dy}{y}.$$

In fact if $f \in C_c^\infty(\mathbb{R})$, then $Hf(x)$ is the pointwise limit of $H_\varepsilon f(x)$ as $\varepsilon \to 0$, whereas the above assertion implies that $H_\varepsilon f \to Q_0 f$ in $L^2(\mathbb{R})$. To prove the above assertion, one simply observes that

$$H_\varepsilon f - f * Q_\varepsilon = f * \Delta_\varepsilon$$

for some kernel Δ_ε, where

$$\Delta_\varepsilon(x) = \varepsilon^{-1} \Delta_1(x/\varepsilon), \quad \Delta_1 \in L^1,$$

and

$$\int_{-\infty}^{\infty} \Delta_1(x) dx = 0.$$

In fact

$$\Delta_1(x) = \begin{cases} \frac{1}{\pi x} - Q_1(x) & \text{if } |x| \geq 1 \\ -Q_1(x) & \text{if } |x| < 1 \end{cases}$$

is odd and satisfies

$$|\Delta_1(x)| \leq \frac{A}{1+x^2}.$$

As a result,

$$H_\varepsilon f(x) - f * Q_\varepsilon(x) = \int_{-\infty}^{\infty} (f(x - \varepsilon t) - f(x)) \Delta_1(t) dt,$$

from which it follows that

$$\|H_\varepsilon f - f * Q_\varepsilon\|_{L^2} \leq \int_{-\infty}^{\infty} \|f(x - \varepsilon t) - f(x)\|_{L^2} |\Delta_1(t)| dt \to 0$$

as $\varepsilon \to 0$.

Incidentally, the above identifies the Hilbert transform Hf of f as the boundary value of the conjugate harmonic extension $f * Q_y$ of f, and as half of the imaginary part of the Cauchy integral Cf if f is real-valued.

6. L^2 THEORY: ALMOST ORTHOGONALITY

If a singular integral operator is given by convolution against a tempered distribution, then one can prove its L^2 boundedness using the Fourier transform. On the other hand, there are naturally arising singular integrals that satisfies the conditions of Theorem 3 but that are not given by convolutions. One is given, for instance, by Cauchy integrals over Lipschitz curves. To introduce this, let γ be a Lipschitz curve given by

$$\gamma = \{x + iA(x) \in \mathbb{C} \colon x \in \mathbb{R}\},$$

where A is a Lipschitz function on $\mathbb{R}$. The Cauchy integral on γ is defined by

$$C_\gamma f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \notin \gamma,$$

if $f \in L^2$ on γ. To study the boundary values of the Cauchy integral on γ, we introduce the Hilbert transform along γ, defined initially for $f \in C_c^\infty(\mathbb{R})$ by

$$H_\gamma f(x) = \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)(1 + iA'(t))}{x - t + i(A(x) - A(t))} dt.$$

If $A \equiv 0$ this reduces to the ordinary Hilbert transform.

Theorem 4. *If γ is as above, then H_γ extends to a bounded linear operator on $L^p(\mathbb{R})$ for all $1 < p < \infty$.*

The crux of this theorem is in its L^2 theory. In fact the L^p theory follows from Theorem 3 once one shows that H_γ extends to a bounded linear operator on $L^2(\mathbb{R})$. To tackle the L^2 theory of H_γ, one has to go beyond the Fourier transform. This is actually the starting point of a long story. Today we will only contend ourselves to proving the following almost orthogonality lemma of Cotlar-Stein, which is the first step of this long journey:

Theorem 5 (Cotlar-Stein)**.** *Suppose V is a Hilbert space and $\{T_j\}_{j \in \mathbb{Z}}$ is a sequence of bounded linear operators on V such that*

$$\|T_j T_k^*\| + \|T_j^* T_k\| \leq \gamma_{j-k}^2 \quad \text{for all } j, k \in \mathbb{Z},$$

where $\{\gamma_j\}_{j \in \mathbb{Z}}$ is a sequence of non-negative real numbers such that $\sum_j \gamma_j \leq A < \infty$. Then the operator $\sum_{j=-N}^N T_j$ converges strongly to some bounded linear operator T on V as $N \to \infty$, i.e. there exists a bounded linear operator T on V such that $\sum_{j=-N}^N T_j f \to Tf$ for all $f \in V$. In fact

$$\|T\| \leq A.$$

The proof is in several steps. First we have the following simple observations: if T is a bounded linear map on V, then

$$\|T\| = \sup_{\|x\|,\|y\| \leq 1} (Tx, y).$$

Hence

$$\|T\| = \|T^*\|,$$

and

$$\|T^*T\| = \|T\|^2 = \|TT^*\|.$$

In particular, if T is a self-adjoint bounded linear operator on V, then $\|T^2\| = \|T\|^2$, and more generally

$$\|T^m\| = \|T\|^m$$

for any $m \in \mathbb{N}$ that is a power of 2.

Now suppose T_j are as in the statement of the theorem. We first prove that for every $N \in \mathbb{N}$, the sum $S_N := \sum_{|j| \leq N} T_j$ is bounded on V with norm $\leq A$. To prove that, note that for any $m \in \mathbb{N}$ that is a power of 2, we have

$$\|S_N\|^{2m} = \|S_N S_N^*\|^m = \|(S_N^* S_N)^m\|.$$

But

$$(S_N^* S_N)^m = \sum_{i_1,\dots,i_{2m}} T_{i_1}^* T_{i_2} \dots T_{i_{2m-1}}^* T_{i_{2m}}.$$

We estimate the operator norm of this sum by estimating the operator norm of each term.

First, writing the summand above as $(T_{i_1}^* T_{i_2}) \dots (T_{i_{2m-1}}^* T_{i_{2m}})$, we get

$$\|T_{i_1}^* T_{i_2} \dots T_{i_{2m-1}}^* T_{i_{2m}}\| \leq \gamma_{i_1-i_2}^2 \gamma_{i_3-i_4}^2 \dots \gamma_{i_{2m-1}-i_{2m}}^2.$$

On the other hand, the same summand above can also be grouped as $T_{i_1}^* (T_{i_2} T_{i_3}^*) \dots (T_{i_{2m-2}} T_{i_{2m-1}}^*) T_{i_{2m}}$, from which we have

$$\|T_{i_1}^* T_{i_2} \dots T_{i_{2m-1}}^* T_{i_{2m}}\| \leq A^2 \gamma_{i_2-i_3}^2 \dots \gamma_{i_{2m-2}-i_{2m-1}}^2,$$

since $\|T_{i_1}^*\|$ and $\|T_{i_{2m}}\|$ are bounded by A. (In fact, $\|T_{i_1}^*\|^2 = \|T_{i_1}^* T_{i_1}\| \leq \gamma_0^2 \leq A^2$, and similarly for $\|T_{i_{2m}}\|$.) Hence taking the geometric mean of these two estimates, we get

$$\|T_{i_1}^* T_{i_2} \dots T_{i_{2m-1}}^* T_{i_{2m}}\| \leq A \gamma_{i_1-i_2} \gamma_{i_2-i_3} \dots \gamma_{i_{2m-1}-i_{2m}}.$$

Summing over $i_1, \dots, i_{2m}$, we get

$$\|S_N\|^{2m} \leq A \sum_{i_1,\dots,i_{2m}} \gamma_{i_1-i_2} \gamma_{i_2-i_3} \dots \gamma_{i_{2m-1}-i_{2m}} \leq A^{2m} N.$$

Taking $2m$-th root, we get

$$\|S_N\| \leq A N^{1/2m}.$$

Since this is true for all $m \in \mathbb{N}$ that are powers of 2, letting $m \to \infty$, we get

$$\|S_N\| \leq A$$

independent of N. This completes the proof of our first step.

Next, we invoke the following lemma:

Lemma 6. *Suppose $\{f_j\}_{j \in \mathbb{Z}}$ is a sequence of vectors in a Hilbert space. Suppose also that for any choice of scalars $\{\varepsilon_j\}$ with $|\varepsilon_j| \leq 1$ for all j and $\varepsilon_j \neq 0$ for all but finitely many j's, we have $\| \sum_j \varepsilon_j f_j \| \leq A$, for some fixed A. Then $\sum_{|j| \leq N} f_j$ converges in norm as $N \to \infty$.*

From this lemma, and what we have proved above, we see that S_N converges strongly to some bounded linear operator T as $N \to \infty$. In fact, for every $f \in V$, if $f_j := T_j f$, then by the first step of our proof, the hypothesis of the lemma is satisfied. Thus $S_N f = \sum_{|j| \leq N} T_j f$ converges in norm to some element in V. Let's call this element Tf. Then T is a linear operator on V, with

$$\|Tf\| = \lim_{N \to \infty} \|S_N f\| \leq A \|f\|.$$

Thus the proof of the theorem will be complete once we prove the lemma.

But the proof of the lemma is rather easy. We argue by contradiction. Suppose $\{f_j\}_{j \in \mathbb{Z}}$ is as in the lemma, but $\sum_{|j| \leq N} f_j$ does not converge in V. Then there exists $\delta > 0$ and $N_1 < N_1' < N_2 < N_2' < \dots$ such that $\Delta_k := \sum_{N_k < |j| \leq N_k'} f_j$ satisfies $\|\Delta_k\| \geq \delta$ for all $k = 1, 2, \dots$. Now the hypothesis implies

$$\left\| \sum_{k=1}^{K} \varepsilon_k \Delta_k \right\|^2 \leq A^2$$

for all choices of $\varepsilon_k \in (-1, 1)$ and all $K \in \mathbb{N}$. Averaging over the ε_k's for $k = 1, \dots, K$, we get

$$\sum_{k=1}^{K} \|\Delta_k\|^2 \leq A^2$$

for all $K \in \mathbb{N}$. This cannot possibly be true by our lower bound on $\|\Delta_k\|$. Thus we arrive at the desired contradiction.