FOURIER ANALYSIS ON FINITE GROUPS (AND OTHER NICE GROUPS)

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1. Introduction

One of the most useful tools in mathematical analysis is the *Fourier series* of a periodic function, which decomposes the given function into a sum of simple oscillatory functions. The classical case deals with an L-periodic function f on $\mathbb{R}$ and its Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x/L},$$

where the *nth Fourier coefficient* $\hat{f}(n)$ is defined to be

$$\hat{f}(n) = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-2\pi i n x/L} \, dx \tag{1}$$

for each $n \in \mathbb{Z}$. An elegant theory emerges when we restrict our attention to square-integrable functions on $[-L/2, L/2]$, the collection of which is a Hilbert space, denoted by $L^2([-L/2, L/2])$, with the inner product

$$\langle f, g \rangle_{L^2([-L/2,L/2])} = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \overline{g(x)} \, dx$$

and the associated L^2 norm

$$\|f\|_{L^2([-L/2,L/2])} = \left(\frac{1}{L} \int_{-L/2}^{L/2} |f(x)|^2 \, dx \right)^{1/2}.$$

Indeed, it can be shown that the Fourier series of an arbitrary square-integrable function converges to the function in the L^2 norm, and that the Fourier basis $\{e^{2\pi i n x/L} : n \in \mathbb{Z}\}$ is, in a precise sense, better than any other orthonormal (Hilbert) basis in $L^2([-L/2, L/2])$.

The classical theory of Fourier analysis also deals with nonperiodic functions, which can be thought of as the limiting case of L-periodic functions as L tends to infinity. Indeed, we make a simple change of variable in (1) to obtain

$$\hat{f}(n) = \int_{-1/2}^{1/2} f(Lx) e^{-2\pi i n x} \, dx.$$

By "sending L to infinity", we are led to the following integral:

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} \, dx.$$

Since the integral makes sense even when n is not an integer, we replace n with a real variable ξ:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx.$$

This is the *Fourier transform* of a complex-valued function f on $\mathbb{R}$, which makes sense if f is integrable on $\mathbb{R}$. The L^2 theory extends to the nonperiodic case as well, with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

and the associated L^2 norm

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^{1/2}.$$

With this setting, it can be shown that the integral given by the *Fourier inversion formula*

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-2\pi ix\xi}\,d\xi$$

converges to f in the L^2 norm.

Now, we might wonder in what sense the theory of Fourier series is "the same" as the theory of Fourier transform on $\mathbb{R}$. The analogy becomes clearer once we consider L-periodic functions on $\mathbb{R}$ as functions on the circle group

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

via the change-of-variable map $x \mapsto e^{2\pi ix/L}$. Indeed, both $\mathbb{T}$ and $\mathbb{R}$ are topological groups, viz., the group operation and the inverse operation are continuous under their topologies. Furthermore, the group is abelian, and its topology is locally compact. We also observe that integration on each group is translation-invariant, in the sense that

$$\int f(x+h)\,dx = \int f(x)\,dx$$

for any real number h. In other words, both groups are equipped with *Haar measures*—translation-invariant Borel regular measures. Perhaps there is a general theory of Fourier analysis on such groups.

2. Fourier Analysis on Finite Cyclic Groups

The goal of the talk is to sketch the theory of Fourier analysis on locally compact abelian groups equipped with Haar measures. This, of course, sounds awfully complicated, so we make some simplifying assumptions. Indeed, we shall focus on finite groups for the majority of the talk.

What benefit might this bring? For one, any topology on a finite group that is compatible with the group structure must be compact and discrete. Therefore, every function on the group is continuous, every convergent sequence is eventually constant, every real-valued function attains a maximum and a minimum, and so on. In fact, we may as well forget about the topology entirely, as it is as nice as it could possibly be.

Furthermore, any nontrivial measure must assign a positive measure to each singleton, and the translation invariance forces every singleton to be of the same measure. Therefore, every function on a finite group equipped a Haar measure is integrable, and the measure of a subset of the group is little more than a constant multiple of the cardinality of the subset. This drop measure and integration theory from our discussion. The sticky technical issues of real analysis disappear entirely, thus allowing us to focus on the big picture.

We shall begin by considering cyclic groups, for the unit circle, on which the theory of Fourier series is developed, can be approximated by regular N-gons as N tends to infinity. Since the cyclic group of order N is isomorphic to the additive group $\mathbb{Z}/N\mathbb{Z}$ of integers modulo N, we shall use the notations from modular arithmetic. Without further ado, we define the function space we shall be working on:

Definition 1. The *Lebesgue space on the cyclic group of order* N, denoted by $L(\mathbb{Z}/N\mathbb{Z})$, is the collection of all complex-valued functions on $\mathbb{Z}/N\mathbb{Z}$. The inner product on $L(\mathbb{Z}/N\mathbb{Z})$ is defined to be

$$\langle f, g\rangle = \frac{1}{N}\sum_{n\in\mathbb{Z}/N\mathbb{Z}} f(n)\overline{g(n)}$$

for each $f, g \in L(\mathbb{Z}/N\mathbb{Z})$, and the associated norm is defined to be

$$\|f\| = \left(\frac{1}{N}\sum_{n\in\mathbb{Z}/N\mathbb{Z}} |f(n)|^2\right)^{1/2}$$

for each $f \in L(\mathbb{Z}/N\mathbb{Z})$.

It is easy to see that $L(\mathbb{Z}/N\mathbb{Z})$ is an N-dimensional complex vector space, and that $\langle , \rangle$ is an inner product.

To develop a theory of Fourier analysis on $\mathbb{Z}/N\mathbb{Z}$, we need the analogues of exponential functions in $L(\mathbb{Z}/N\mathbb{Z})$. Imitating the usual Fourier basis in $L^2(\mathbb{T})$, we define

$$e_m(n) = e^{2\pi i mn/N}$$

for each m and every n in $\mathbb{Z}/N\mathbb{Z}$. The expression is well-defined, for $e^{2\pi i N/N} = 1$.

Proposition 2 (Orthogonality relations)**.** $e_1, \ldots, e_N$ *is an orthonormal basis of* $L(\mathbb{Z}/N\mathbb{Z})$.

Proof. We first observe that

$$\langle e_m, e_m\rangle = \frac{1}{N}\sum_{n\in\mathbb{Z}/N\mathbb{Z}} 1 = 1$$

for each m. To show that $\langle e_{m_1}, e_{m_2}\rangle = 0$ for any $m_1 \neq m_2$, we recall that the sum of all nth roots of unity is 0. Indeed, if ω_n is the primitive nth root of unity, then

$$0 = \omega_n^n - 1 = (\omega_n - 1)\sum_{k=0}^{n-1}\omega_n^k.$$

Since $\omega_n \neq 1$, it follows that

$$\sum_{k=0}^{n-1}\omega_n^k = 0.$$

We now fix two distinct elements $m_1, m_2 \in \mathbb{Z}/N\mathbb{Z}$ and define m to be the greatest common divisor of $m_1 - m_2$ and N. Setting $M = N/m$, we see that a simple rescaling argument yields

$$\langle e_{m_1}, e_{m_2} \rangle = \frac{1}{N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} e^{2\pi i (m_1 - m_2) n/N} = \frac{1}{N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} e^{2\pi i n/M}.$$

The latter sum consists of m copies of the sum of all Mth roots of unity, whence it equals zero. This completes the proof. □

Imitating the usual definition once again, we define the cyclic analogue of the Fourier series:

Definition 3. The *discrete Fourier transform* of $f \in L(\mathbb{Z}/N\mathbb{Z})$ is the function $\hat{f} \in L(\mathbb{Z}/N\mathbb{Z})$ defined at each $m \in \mathbb{Z}/N\mathbb{Z}$ to be

$$\hat{f}(m) = \langle f, e_m \rangle.$$

The *discrete Fourier series* of f is defined to be the finite sum

$$\sum_{m \in \mathbb{Z}/N\mathbb{Z}} \hat{f}(m) e_m. \tag{2}$$

It is a standard result in linear algebra that (2) is the correct expansion in the orthonormal basis $e_1, \ldots, e_N$. Therefore, we have the *Fourier inversion formula*

$$f(n) = \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \hat{f}(m) e_m(n)$$

for all $f \in L(\mathbb{Z}/N\mathbb{Z})$. Therefore, the *discrete Fourier transform operator* $\mathscr{F} : L(\mathbb{Z}/N\mathbb{Z}) \to L(\mathbb{Z}/N\mathbb{Z})$ defined by the mapping

$$f \mapsto \hat{f}$$

is invertible. Clearly, $\mathscr{F}$ is a linear transformation, whence $\mathscr{F}$ is a linear automorphism on $L(\mathbb{Z}/N\mathbb{Z})$. In this way, the discrete Fourier transform is similar to the nonperiodic Fourier transform, which transforms functions on $\mathbb{R}$ to functions on $\mathbb{R}$.

Pushing the analogy further, we now establish the following theorem, whose continuous analogue is a cornerstone of continuous Fourier analysis:

Theorem 4 (Plancherel's theorem)**.** *$\mathscr{F}$ is an "isometry[1]" on $L(\mathbb{Z}/N\mathbb{Z})$, viz.,*

$$\|\hat{f}\| = N^{-1/2} \|f\|$$

for all $f \in L(\mathbb{Z}/N\mathbb{Z})$.

[1]We could, of course, renormalize the inner product or the Fourier transform to turn this into an isometry. Doing so would result in the loss of orthonormal basis status of the Fourier basis, which we prefer to have in our exposition.

Proof. By the Fourier inversion formula, we have

$$\begin{aligned}\|f\|^2 &= \langle f, f\rangle \\ &= \left\langle \sum_{m\in\mathbb{Z}/N\mathbb{Z}} \hat{f}(m)e_m, \sum_{m'\in\mathbb{Z}/N\mathbb{Z}} \hat{f}(m')e_{m'} \right\rangle \\ &= \sum_{m\in\mathbb{Z}/N\mathbb{Z}} \sum_{m'\in\mathbb{Z}/N\mathbb{Z}} \hat{f}(m)\overline{\hat{f}(m')}\langle e_m, e'_m\rangle \\ &= N\cdot\frac{1}{N}\sum_{m\in\mathbb{Z}/N\mathbb{Z}} |\hat{f}(m)|^2 \\ &= N\|\hat{f}\|^2,\end{aligned}$$

as was to be shown. □

Just as in the continuous case, the discrete Fourier transform is translation-invariant. The trivial proof of the following proposition is omitted.

Proposition 5. *For each $f \in L(\mathbb{Z}/N\mathbb{Z})$ and $n_0 \in \mathbb{Z}/N\mathbb{Z}$, we define the* translation operator $\mathscr{T}_{n_0}$ *by setting*

$$\mathscr{T}_{n_0} f(n) = f(n - n_0).$$

With this definition, we have

$$\widehat{T_{n_0} f}(m) = e^{-2\pi i n_0 m/N}\hat{f}(m).$$

3. Application: The Discrete Isoperimetric Inequality

Having developed basic Fourier analysis on $\mathbb{Z}/N\mathbb{Z}$, we now consider an application. Recall the *isoperimetric inequality* for curves in the plane, which states that every rectifiable closed simple curve in the plane with length L and interior area A satisfies the inequality

$$A \leq \frac{L^2}{4\pi},$$

whose equality is attained if and only if the curve is a circle. Our goal is to prove the following discrete analogue:

Theorem 6 (Discrete isoperimetric inequality)**.** *If $z : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ defines a closed equilateral polygon $\{z(0), z(1), \ldots, z(N-1)\}$ centered at the origin in the plane with length L and interior area A, then*

$$\frac{A}{L^2} \leq \frac{1}{4N}\cot\frac{\pi}{N}$$

To this end, we first establish the following formula:

Lemma 7. *If z is defined as in Theorem 6, then*

$$A = \frac{N}{2}\sum_{m=0}^{N-1} |\hat{z}(m)|^2 \sin\frac{2\pi m}{N}.$$

Proof. For each $0 \leq n \leq N-1$, we write

$$z(n) = x(n) + iy(n),$$

where x and y are real-valued functions on $\mathbb{Z}/N\mathbb{Z}$. We observe that the area of the triangle $Oz(n)z(n+1)$ can be written as

$$\frac{1}{2}\left[x(n)y(n+1) - x(n+1)y(n)\right] = \frac{1}{2}\operatorname{Im}\left(\overline{z}(n)z(n+1)\right).$$

Since the area of the polygon is the sum of these N triangles, we have

$$A = \frac{1}{2}\sum_{n=0}^{N-1}\operatorname{Im}\left(\overline{z}(n)z(n+1)\right) = \frac{1}{4i}\sum_{n=0}^{N-1}\overline{z}(n)z(n+1) - z(n)\overline{z}(n+1). \tag{3}$$

The Fourier inversion formula yields

$$\begin{aligned}
z(n) &= \sum_{m=0}^{N-1}\hat{z}(m)e_m(n)\\
\overline{z}(n) &= \sum_{m=0}^{N-1}\overline{\hat{z}}(m)e_{-m}(n)\\
z(n+1) &= \sum_{m=0}^{N-1}e^{2\pi i m/N}\hat{z}(m)e_m(n)\\
\overline{z}(n+1) &= \sum_{m=0}^{N-1}e^{-2\pi i m/N}\hat{z}(m)e_{-m}(n).
\end{aligned}$$

Therefore, the orthogonality relations and (3) imply

$$A = \frac{N}{4i}\sum_{m=0}^{N-1}|\hat{z}(m)|^2(e^{2\pi i m/N} - e^{-2\pi i m/N}) = \frac{N}{2}\sum_{m=0}^{N-1}|\hat{z}(m)|^2\sin\frac{2\pi m}{N},$$

as was to be shown. □

We now proceed to prove the discrete isoperimetric inequality.

Proof of Theorem 6. Since the polygon is equilateral,

$$\frac{L^2}{N} = N\cdot\left(\frac{L}{N}\right)^2 = N\cdot|z(1)-z(0)|^2 = \sum_{n=0}^{N-1}|z(n+1)-z(n)|^2. \tag{4}$$

By Plancherel's theorem and Proposition 5, we have

$$\begin{aligned}
\sum_{n=0}^{N-1}|z(n+1)-z(n)|^2 &= N\sum_{m=0}^{N-1}|\hat{z}(m+1)-\hat{z}(m)|^2\\
&= N\sum_{m=0}^{N-1}|e^{2\pi i m/N}-1|^2|\hat{z}(m)|^2\\
&= 4N\sum_{m=0}^{N-1}\left|\sin\frac{\pi m}{k}\right|^2|\hat{z}(m)|^2,
\end{aligned}$$

whence (4) implies that

$$L^2 = (2N)^2\sum_{m=0}^{N-1}\left|\sin\frac{\pi m}{k}\right|^2|\hat{z}(m)|^2. \tag{5}$$

By Lemma 7, we have

$$\begin{aligned} 4NA\tan\frac{\pi}{N} &= \left(2N^2\tan\frac{\pi}{N}\right)\sum_{m=0}^{N-1}|\hat{z}(m)|^2\sin\frac{2\pi m}{N} \\ &= 4N^2\sum_{m=0}^{N-1}|\hat{z}(m)|^2\sin\left(\frac{\pi m}{N}\right)\cos\left(\frac{\pi m}{N}\right)\tan\left(\frac{\pi}{N}\right). \end{aligned}$$

Therefore,

$$L^2 - 4NA\tan\frac{\pi}{N} = 4N^2\sum_{m=0}^{N-1}|\hat{z}(m)|^2\sin\left(\frac{\pi m}{N}\right)\left[\sin\left(\frac{\pi m}{N}\right) - \cos\left(\frac{\pi m}{N}\right)\tan\left(\frac{\pi}{N}\right)\right].$$

We claim that $L^2 - 4NA\tan\frac{\pi}{N} \geq 0$, which then establishes the inequality

$$\frac{A}{L^2} \leq \frac{1}{4N}\cot\frac{\pi}{N}.$$

To show this, it suffices to prove

$$\sin\left(\frac{\pi m}{N}\right)\left[\sin\left(\frac{\pi m}{N}\right) - \cos\left(\frac{\pi m}{N}\right)\tan\left(\frac{\pi}{N}\right)\right] \geq 0$$

for each $0 \leq m \leq N-1$. Furthermore, we have $\sin(\pi m/N) \geq 0$ for all such m, whence it is enough to establish the inequality

$$\sin\left(\frac{\pi m}{N}\right) - \cos\left(\frac{\pi m}{N}\right)\tan\left(\frac{\pi}{N}\right) \geq 0,$$

But the above is equivalent to

$$\tan\frac{\pi m}{N} \geq \tan\frac{\pi}{N},$$

which holds for all such m. Therefore, the claim holds, and the proof is now complete. □

We remark that the discrete isoperimetric inequality achieves the equality if and only if the polygon at hand is regular.

4. Concluding Remarks: Abstract Harmonic Analysis

We have alluded to the theory of Fourier analysis on locally compact abelian groups with Haar measures. This is the subject of *abstract harmonic analysis*, which we now touch upon briefly.

Recall that a *topological group* is a group G with a topology such that the group-operation map $(x, y) \mapsto xy$ on $G \times G$ into G and the inverse map $x \mapsto x^{-1}$ on G into itself are continuous. A basic theorem in abstract harmonic analysis guarantees that every locally compact Hausdorff abelian group G admits a *Haar measure*, which is a nonzero Borel measure μ on G such that

(1) $\mu(E) = \mu(gE) = \mu(Eg)$ for each $g \in G$ and every Borel set $E \subseteq G$.
(2) For each Borel subset E of G, the measure of E is the supremum of the measure of compact subsets of E;
(3) Every point in G has a neighborhood of finite measure.

Furthermore, all Haar measures on G are constant multiples of one another, whence we can speak of "the" Haar measure on G, by stipulating the measure of the whole group to be 1.

How might we develop "Fourier analysis" on G? Going back to finite cyclic groups, we observe that each $e_m(n)$ is a group homomorphism on $\mathbb{Z}/N\mathbb{Z}$ into the circle group $\mathbb{T}$. Therefore, we can consider the discrete Fourier transform as an act of describing functions on $\mathbb{Z}/N\mathbb{Z}$ in terms of characters on $\mathbb{Z}/N\mathbb{Z}$. Therefore, we should expect that Fourier analysis on G requires us to study the characters on G.

Formally, a *unitary character* on G is a topological group morphism—a group homomorphism that is continuous—on G into $\mathbb{T}$. The collection $\hat{G}$ of unitary characters on G forms a group, called the *dual group* of G. Generalizing the notion of "rewriting the function in the Fourier space" from continuous Fourier analysis, we should expect the Fourier transform on G to transform a function on G to a function on $\hat{G}$: $\hat{G}$ shall be, in some sense, our "Fourier basis". Indeed, if G is compact, then $\hat{G}$ is an orthonormal set in $L^2(G)$.

The *Fourier transform* of $f \in L^1(G)$ is defined to be

$$(\mathscr{F} f)(\xi) = \hat{f}(\xi) = \int f(x)\overline{\xi(x)}\,dx$$

for each $\xi \in \hat{G}$. The basic theorems of Fourier analysis, such as the Hausdorff-Young inequality, the Fourier inversion formula, and Planceherel's theorem continue to hold in this setting. Furthermore, the *Pontrayagin duality theorem* states that the double dual $\hat{\hat{G}}$ of G is isomorphic to G, whereby we recover the standard duality relation between the "physical space" and the "Fourier space".

Abstract harmonic analysis is at the intersection of many disciplines of mathematics. To see this, let us examine the structure of $\hat{G}$ in greater detail. The natural topology on $\hat{G}$ is the *compact-open topology.* This coincides with the weak-* topology on G, which is the weak topology induced by the canonical embedding $G \hookrightarrow \hat{\hat{G}}$. Under this topology, $\hat{G}$ is a locally compact topological group. Furthermore, the Fourier transform $\mathscr{F}$ can be thought of as an operator on $L^1(G)$ into the space $\mathscr{C}(\hat{G})$ Viewing $L^1(G)$ as a *-algebra with convolution as the product, we can prove that $\mathscr{F}$ is the Gelfand transform on $L^1(G)$. Indeed, $\mathscr{F}$ is a norm-decreasing *-homomorphism into the sapce $\mathscr{C}_0(G)$ of continuous maps on G vanishing at infinity. Pursuing this viewpoint further, we can identify $\hat{G}$ with the spectrum of $L^1(G)$ via the *-representation

$$\xi(f) = \int f(x)\xi(x)\,dx,$$

and the closed ideals in $L^1(G)$ can be studied in great detail.

We can also consider *unitary representations* of G, which are homomorphisms π from G into the group $U(\mathscr{H}_\pi)$ of unitary operators on nonempty Hilbert space $\mathscr{H}_\pi$ that are continuous with respect to the strong operator topology. In this setting, $\mathscr{H}_\pi$ is called the *representation space* of π, and $\dim \mathscr{H}_\pi$ the *degree* of π. A topic of interest in representation theory is the study of irreducible representations, which is defined as follows:

(a) A closed subspace $\mathscr{M}$ of $\mathscr{H}_\pi$ is *invariant* if $\pi(x)\mathscr{M} \subseteq \mathscr{M}$ for all $x \in G$.
(b) The restriction of π on an invariant subspace is a *subrepresentation* of π. The subrepresentation is *proper* if it is a restriction on a proper subspace.
(c) π is *irreducible* if π is not reducible.

It can be shown that all irreducible representations of G are one-dimensional, whence $\hat{G}$ *is* the collection of irreducible representations of G. Therefore, the representation theory of locally compact abelian group is, in a sense, generalized harmonic analysis. This also provides connections between harmonic analysis and differential geometry, where Lie groups play an important role.

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