KITTENS

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Abstract. 

In $\mathbb{R}^3$, we define the *Laplacian* $\Delta f$of f as

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f).$$

Recall also that

$$\text{0-forms} \underset{\overset{d}{\to}}{\xrightarrow{\text{grad}}} \text{1-forms} \underset{\overset{d}{\to}}{\xrightarrow{\text{curl}}} \text{2-forms} \underset{\overset{d}{\to}}{\xrightarrow{\text{div}}} \text{3-forms}$$

We shall define the *Hodge star-operator* $* : \Lambda^p V \to \Lambda^{n-p} V$. To do so, we need an orientation and an inner product. To this end, we take an orthonormal basis $\{e_1, \dots, e_n\}$ for V, so that $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$ is an orthonormal basis for $\Lambda^p V$. We now define

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots e_{j_{n-p}}$$

such that $e_{i_1}, \dots, e_{i_p}, e_{j_1}, \dots, e_{j_{n-p}}$ is positively oriented. We note that

$$** = (-1)^{p(n-p)}.$$

Given a function $f(X, y, z)$, we observe that

$$\begin{aligned}
d(df) &= *d*(f_x\,dx + f_y\,dy + f_z\,dz) \\
&= *d(f_x\,dy\,dz - f_y\,dx\,dz + f_z\,dx\,dy) \\
&= *(f_{xx}\,dx\,dy\,dz + f_{yy}\,dx\,dy\,dz + f_{zz}\,dx\,dy\,dz) \\
&= f_{xx} + f_{yy} + f_{zz} \\
&= \Delta f.
\end{aligned}$$

It follows that $\Delta = (*d*)d + d(*d*)$. In general, if M is an oriented Riemannian manifold, then $* : \Lambda^p T_x^* M \to \Lambda^{n-p} T_x^* M$ and $* : \Omega^p(M) \to \Omega^{n-p}(M)$.

We now let M be an oriented compact Riemannian manifold. For each $\alpha, \beta \in \Omega^p$, we set

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

and observe that

$$\begin{aligned} \langle d\alpha, \beta \rangle &= \langle \alpha, \delta\beta \rangle \\ \delta &= (-1)^{n(p+1)+1} * d * . \end{aligned}$$

The *Laplace-Beltrami operator* on M is defined by

$$\Delta = d\delta + \delta d.$$

Note that Δ is a self-adjoint operator.

Let us now turn to differential equations. Generalizing the Euclidean definition, we say that a p-form ω is *harmonic* if $\Delta\omega = 0$. The following theorem provides a necessary and sufficient condition for harmonicity of p-forms.

Theorem 1. *$\Delta\omega = 0$ if and only if $d\omega = 0$ and $\delta\omega = 0$.*

Proof. If $\Delta\omega = 0$, then

$$\begin{aligned} \langle \Delta\omega, \omega \rangle &= \langle (d\delta + \delta d)w, w \rangle \\ &= \langle d\delta\omega, \omega \rangle + \langle \delta d\omega, \omega \rangle \\ &= \langle \delta w, \delta w \rangle + \langle d\omega, d\omega \rangle \\ &= 0. \end{aligned}$$

□

We now consider Poisson's equation

$$\Delta w = \alpha$$

First, let us try to find the weak solutions. To this end, we suppose that ω is a solution and observe that

$$\langle \Delta\omega, \beta \rangle = \langle \alpha, \beta \rangle = \langle \omega, \Delta\beta \rangle$$

for all $\beta \in \Omega^p$. The linear functional φ on Ω^p defined by $\varphi(\beta) = \langle \omega, \beta \rangle$ satisfies the identity

$$\varphi(\Delta\beta) = \langle \alpha, \beta \rangle,$$

whence we refer to any such linear functional as a *weak solution*. As it turns out, finding such a linear functional is a nontrivial task, and the proof of the existence theorem below shall be omitted.

Theorem 2. *Let $\alpha \in \Omega^p$ φ a weak solution of $\Delta\omega = \alpha$. In this case, there exists an $\omega \in \Omega^p$ such that $\varphi(\beta) = \langle \omega, \beta \rangle$ for $\beta \in \Omega^p$, so that $\Delta\omega = \alpha$.*

Theorem 3. *If $(a_n)_{n=1}^\infty$ is a sequence in Ω^p such that $\|\alpha_n\|$ and $\|\Delta\alpha_n\|$ are uniformly bounded in n, then we can extract a Cauchy subsequence of $(a_n)_{n=1}^\infty$.*

Henceforth, we shall write H^p to denote the space of harmonic p-forms on an oriented compact Riemannian manifold M. Here is the main theorem of the talk:

Theorem 4 (Hodge decomposition theorem)**.** *Let M be an oriented comapct Riemannian manifold. For each integer $p \in [0,n]$, we have the following orthogonal dedcomposition:*

$$\Omega^p(M) = \Delta(\Omega^p) \oplus H^p.$$

It then follows that Poisson's equation $\Delta\omega = \alpha$ has a solution if and only if $\alpha \in (H^p)^\perp$.

Proof. We first show that H^p is finite-dimensional. To this end, we suppose for a contradiction that there eixsts a collection $\{\alpha_n : n \in \mathbb{N}\}$ of orthonormal harmonic forms, viz., $\|\alpha_n\| = 1$ and $\|\Delta\alpha_n\| = 0$. By Theorem 3, there is a Cauchy subsequence of this collection. We, however, observe that

$$\begin{aligned} \|\alpha_n - \alpha_m\|^2 &= \langle \alpha_n - \alpha_m, \alpha_n - \alpha_m \rangle \\ &= \langle \alpha_n, \alpha_n \rangle - 2\langle \alpha_n, \alpha_m \rangle + \langle \alpha_m, \alpha_m \rangle \\ &= 1 + 0 + 1 \\ &= 2, \end{aligned}$$

which contradicts the existence of a Cauchy subsequence. It follows that H^p is fiite-dimensional.

We now let $\omega_1, \ldots, \omega_m$ be an orthonormal basis of H^p. Given $\alpha \in \omega^p$, we write

$$\alpha = \beta + \sum \langle \alpha, \omega_k \rangle \omega_k$$

such that $\beta \in (H^p)^\perp$. We would like to show that $(H^p)^\perp = \Delta(\Omega^p)$. To this end, we let $H(\alpha)$ be the projection onto H^p. If $\omega \in \Omega^p$ and $\alpha \in H^p$, then

$$\langle \Delta\omega, \alpha \rangle = \langle \omega, \Delta\alpha \rangle = 0.$$

We claim that there exists a constant $c > 0$ such that

$$\|\beta\| \leq c\|\Delta\beta\|$$

for all $\beta \in (H^p)^\perp$.

To establish the claim, we suppose for a contradiction that there exists a sequence $(\beta_j)_{j=1}^\infty$ such that $\|\beta_j\| = 1$ and $\|\Delta\beta_j\| \to 0$ as $j \to \infty$. Here we note that $\beta_j \in (H^p)^\perp$. By Theorem 3, we can extract a Cauchy subsequence of $(\beta_j)_{j=1}^\infty$, whence we may assume without loss of generality that $(\beta_j)_{j=1}^\infty$ is Cauchy.

Define

$$\phi(\gamma) = \lim_{j\to\infty} \langle \beta_j, \gamma \rangle.$$

Note that ϕ is bounded and satisfies the identity

$$\phi(\Delta\gamma) = \lim_{j\to\infty} \langle \beta_j, \Delta\gamma \rangle = \lim_{j\to\infty} \langle \Delta\beta_j, \gamma \rangle = 0.$$

It follows that ϕ is a weak solution of Laplace's equation $\Delta\beta = 0$.

We now invoke Theorem 2 to find a $\beta \in \Omega^p$ such that

$$\phi(\gamma) = \langle \beta, \gamma \rangle$$

for all $\gamma \in \Omega^p$. This, in particular, implies that $\beta_j \to \beta$. Since $\|\beta_j\| = 1$ for all j, we have $\|\beta\| = 1$. Furthermore, $\beta_j \in (H^p)^\perp$ for each j, and so $\beta \in (H^p)^\perp$. We, however, recall that $\Delta\beta = 0$, which implies that $\beta \in H^p$. This is evidently absurd, whence we can find a constant $c > 0$ such that

$$\|\beta\| \leq c\|\Delta\beta\|$$

for all $\beta \in (H^p)^\perp$.

We now define φ on $\Delta(\Omega^p)$ by

$$\varphi(\Delta\beta) = \langle \alpha, \beta \rangle$$

for all $\beta \in \Omega^p$ and $\alpha \in (H^p)^\perp$. If $\Delta\beta_1 = \Delta\beta_2$, then $\Delta(\beta_1 - \beta_2) = 0$, and so

$$\langle \alpha, \beta_1 - \beta_2 \rangle = 0,$$

which shows that φ is well-defined.

Fix $\beta \in \Omega^p$ and let $\gamma = \beta - H(\beta)$. We observe that

$$\begin{aligned} |\varphi(\Delta\beta)| &= |\varphi(\Delta\gamma)| \\ &= |\langle \alpha, \gamma \rangle| \\ &\leq \|\alpha\| \cdot \|\gamma\| \\ &\leq C\|\alpha\|\|\Delta\gamma\| \\ &= C\|\alpha\|\|\Delta\beta\|. \end{aligned}$$

It follows that φ is bounded on $\Delta(\Omega^p)$. We now invoke the Hahn-Banach theorem to extend φ onto all of Ω^p, so that φ is a weak solution of Poisson's equation $\Delta\omega = \alpha$. Theorem 2 now implies that there is an $\omega \in \Omega^p$ such that $\Delta\omega = \alpha$. Therefore, $(H^p)^\perp \subseteq \Delta(\Omega^p)$, and so

$$(H^p)^\perp = \Delta(\Omega^p),$$

as was to be shown. □

Recall now that $d : \Omega^p \to \Omega^{p+1}$ acts on the chain of forms as follows:

$$\cdots \to \Omega^{p-1} \xrightarrow{d_{p-1}} \Omega^p \xrightarrow{d_p} \Omega^{p+1} \xrightarrow{d} \xrightarrow{d_{p+1}} \to \cdots$$

We define the *pth de Rham cohomology* as

$$H^p_{\text{deR}} = \ker(d_p) / \operatorname{im}(d_{p-1})$$

We now have the following application of the main theorem:

Theorem 5. *Every cohomology class in H^p_{deR} contains precisely one harmonic form.*