THE KAKEYA PROBLEM IN HARMONIC ANALYSIS

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ABSTRACT. This is the written version of the talk given at the Rutgers Graduate Student Analysis Seminar on January 30, 2012, following the flow of ideas in [Łab]. Here is the abstract for the talk:

What is the minimum area required to spin a needle around? Yes? Did you say, "That's so 1920s"? Fear not! In this talk, we will present a bird's-eye view of the problems known collectively as the Kakeya problem, ranging from theorems from 1970s to still-unsolved conjectures. The focus will be on the harmonic-analytic approach, though it should be noted that there are plenty of geometry, number theory, and combinatorics floating around in the scene.

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1. INTRODUCTION: BESICOVITCH AND KAKEYA SETS

In 1917, Abram Besicovitch asked the following question:

Question. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is Riemann-integrable, does there exist a pair of orthogonal coordinate axes which would render both $x \mapsto f(x, y)$ and $y \mapsto \int f(x, y)\, dx$ Riemann-integrable?*

Besicovitch noted that a counterexample could be given if a compact set of measure zero in $\mathbb{R}^2$ that contains a line segment in every direction could be constructed. To see this, we assume that F is such a set. We fix a pair of axes and translate F so that the line segments in F parallel to the axes are of irrational distance apart from the axes. Let F_0 be the set of points in F consisting of at least one rational coordinate with respect to the axes chosen. For each line segment l in F, we observe that both $F_0 \cap l$ and $(F \setminus F_0) \cap l$ are dense in l. This results in severe "one-dimensional discontinuity" of the characteristic function χ_{F_0} in each direction, which, in turn, $x \mapsto \chi_{F_0}(x, y)$ is not Riemann-integrable regardless of the choise of axes or y. F, however, is of planar measure zero, whence χ_{F_0} is Riemann-integrable on $\mathbb{R}^2$ by the Lebesgue criterion for Riemann integrability.

Two years later, Besicovitch succeeded in constructing such a set:

Theorem 1 (Besicovitch, 1919)**.** *There exists a compact planar set of measure zero that contains a unit line segment in each direction.*

Roughly, we proceed by considering first an equilateral triangle, slicing it up into thin pieces, and overlapping them in different direction: this yields a set with smaller measure that nevertheless contains a wider variety of unit line segments than the original triangle. Continuing this way, we end up with a set of measure zero that contains a unit line segment in each direction from, say, $0°$ to $90°$; the actual range, of course, depends on the details of the construction. Piecing together several of these sets if necessary, we get the desired set. Terry Tao wrote an interactive applet that shows the first ten steps of the construction[1].

The construct has a straightforward generalization to higher dimensions. For the sake of efficiency, it is convenient to make the following definition:

Definition 2. A *Kakeya set* (or a *Besicovitch set*) in $\mathbb{R}^n$ is a compact set in $\mathbb{R}^n$ that contains a unit line segment in each direction, viz.,

$$\forall e \in \mathbb{S}^{n-1} \quad \exists x \in \mathbb{R}^n \quad \text{s.t. } x + te \in E \quad \forall t \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

By considering the products of the two-dimensional Besicovitch set, we obtain the following result:

Theorem 3 (Besicovitch, 1919)**.** *If $n \geq 2$, then there are Kakeya sets of measure zero in $\mathbb{R}^n$.*

Why the name *Kakeya sets*? Unbeknownst to Besicovitch, the Japanese mathematician Soichi Kakeya was investigating a similar problem in 1917:

Question (Kakeya needle problem)**.** *What is the minimum area of a compact planar set in which a unit line segment can be rotated* $180°$*?*

The civil war and the blockade prevented the rest of the world from finding out about Besicovitch's work at the time. For a number of years the three-cusped hypercycloid was considered to be the optimal solution. Besicovitch eventually learned of the problem, shortly after his departure from Russia in 1924. By slightly modifying his construction of the two-dimensional Kakeya set, Besicovitch was able to establish the following:

Theorem 4 (Besicovitch, 1928)**.** *For each $\varepsilon > 0$, there exists a compact planar set of measure ε in which a unit line segment can be rotated* $180°$*.*

And so the needle problem can be solved on a "very small" set. But just how small is "very small"? This seemingly innocent question is, in fact, a starting point of a deep theory, which serves as a meeting ground for several major areas of modern mathematics. The remainder of the talk will be devoted to sampling some of the cornerstones of the theory.

2. A Quick Overview of the Theory of Lebesgue Spaces

We will have to cover the basic before we begin. As was presented in Math 501, the starting point of modern analysis is the theory of *Lebesgue integration*, presented first by Henri Lebesgue in his immortal 1902 thesis. Eight years later, Hungarian mathematician Frigyes Riesz published a study of functions f with the integrability condition on $|f(x)|^p$, which proved indispensable in all subsequent development of

[1] Alternatively, you could ask Susan Durst to show you the posters she made for her Pizza Seminar talk, which may come equipped with cool sound effects.

analysis. It should thus come as no surprise that the space of such functions bears the name of Lebesgue.

For $p > 0$, the *Lebesgue space* $L^p(\mathbb{R}^n)$ is the space of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

If $1 \leq p < \infty$, then $L^p(\mathbb{R}^n)$ is a normed linear space with the norm $\|\cdot\|_p$ [2]. We also consider $L^\infty(\mathbb{R}^n)$, the space of functions $f : \mathbb{R}^n \to \mathbb{C}$ whose *essential supremum*

$$\|f\|_\infty = \inf\{k \in [0, \infty) : m(\{x \in \mathbb{R}^n : |f(x)| > k\}) = 0\}$$

is finite. We note that $\|\cdot\|_\infty$ is a norm in $L^\infty(\mathbb{R}^n)$. $L^\infty(\mathbb{R}^n)$ is the "limiting case" of $L^p(\mathbb{R}^n)$, in the sense that if f is supported on a set of finite measure, thereby belonging to all $L^p(\mathbb{R}^n)$ for $p < \infty$, then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$. By the *Riesz-Fischer theorem*, the induced metric

$$d_p(f, g) = \|f - g\|_p$$

renders $L^p(\mathbb{R}^n)$ a complete metric space for each $1 \leq p \leq \infty$, thereby turning $L^p(\mathbb{R}^n)$ into a *Banach space*. For simplicity's sake, we shall henceforth drop the $(\mathbb{R}^n)$ and simply write L^p for $L^p(\mathbb{R}^n)$.

Fix $1 \leq p < \infty$, and consider a sequence $(f_n)_{n=1}^\infty$ of functions in L^p. Since L^p is equipped with the norm topology, we could ask whether $(f_n)_{n=1}^\infty$ converges in L^p, and whether L^p-convergence bears any relationship with pointwise convergence. Indeed, the dominated convergence theorem implies that $(f_n)_{n=1}^\infty$ converges to f in L^p if $f_n \to f$ almost everywhere and if there exists a dominating function $g \in L^p$, viz., $|f_n| \leq g$ almost everywhere for all $n \in \mathbb{N}$. Conversely, if $f_n \to f$ in L^p, then we can extract a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ and an L^p function g such that $f_{n_k} \to f$ almost everywhere and $|f_{n_k}| \leq g$ almost everywhere.

How can we transfer convergence in one Lebesgue space to another? Continuous operators would do the job, of course, but the Lebesgue spaces are more than just topological spaces. In fact, they are also vector spaces, whence it is desirable to consider continuous linear operators between them. A convenient characterization is that of *boundedness*, which is defined as follows: a linear operator $T : L^p \to L^q$ is *bounded* if there exists a constant $k > 0$ such that

$$\|Tf\|_q \leq k\|f\|_p$$

for all $f \in L^p$. The smallest such k is the *operator norm* of T, which is denoted by $\|T\|_{L^p \to L^q}$. We note that a linear operator between Lebesgues spaces is continuous if and only if it is bounded.

We need not specify the output for each and every function to define an operator: indeed, if D is a dense subset of $L^p(\mathbb{R}^n)$ and $T : D \to L^q(\mathbb{R}^n)$ a bounded linear operator, then there exists a unique extension $T' : L^p \to L^q$ of T such that $\|T'\|_{L^p \to L^q} = \|T\|_{D \to L^q}$. This, in particular, allows us to make sense of an operator T being simultaneously a bounded operator from L^{p_0} to L^{q_0} and a bounded operator from L^{p_1} to L^{q_1} by considering a set $D \subseteq L^{p_0} \cap L^{p_1}$ that is dense in both L^{p_0} and L^{p_1}. Indeed, if there are constants $k_0, k_1 > 0$ such that

$$\|Tf\|_{q_0} \leq k_1\|f\|_{p_0} \quad \text{and} \quad \|Tf\|_{q_1} \leq k_2\|f\|_{p_1}$$

[2]Technically, $\|f\|_p$ is only a norm "almost everywhere": indeed, we must equate functions that agree with one another almost everywhere to turn $\|\cdot\|_p$ into a norm.

for all $f \in D$, then there are unique extensions $T_1 : L^{p_0} \to L^{q_0}$ and $T_2 : L^{p_1} \to L^{q_1}$ of T such that

$$\|T_1 f\|_{q_0} \leq k_1 \|f\|_{p_0} \quad \text{and} \quad \|T_2 g\|_{q_1} \leq k_2 \|g\|_{p_1}$$

for all $f \in L^{p_1}$ and $g \in L^{p_2}$. In this case, the *Riesz-Thorin interpolation theorem* implies that we have the bound

$$\|T\|_{p_\theta \to q_\theta} \leq \|T\|_{p_0 \to q_0}^{1-\theta} \|T\|_{p_1 \to q_1}^{\theta}$$

for each $\theta \in (0, 1)$, where

$$p_\theta^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1} \quad \text{and} \quad q_\theta^{-1} = (1-\theta)q_0^{-1} + \theta q_1^{-1}.$$

3. The Fourier Inversion Problem

We now restrict our attention to the famous operator of Joseph Fourier. The *Fourier transform* of $f : \mathbb{R}^n \to \mathbb{C}$ at $\xi \in \mathbb{R}^n$ is defined formally to be

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx,$$

where $\cdot$ is the standard dot product in $\mathbb{R}^n$. If $f \in L^1$, then we can make sense of this definition: in fact, $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is continuous, decays at infinity, and satisfies the estimate

$$\|\hat{f}\|_\infty \leq \|f\|_1,$$

which implies that the Fourier transform is a bounded linear operator from L^1 to L^∞.

How about functions in L^p for $1 < p < \infty$? Clearly, the integral definition makes no sense if the function is not in L^1. We can nevertheless define the Fourier transform operator by considering a dense subset of L^p on which the integral definition is valid. The prime candidate is the *Schwartz space* $\mathscr{S}$, which consists of infinitely-differentiable functions on $\mathbb{R}^n$ such that all of their partials, including the functions themselves, decrease more rapidly than polynomials. Since $\mathscr{S}$ is dense in L^p for all $1 \leq p < \infty$, the definition of Fourier transform is valid on $\mathscr{S}$.

In fact, the Fourier transform is a homeomorphism of $\mathscr{S}$ onto itself: the inverse map, called the *Fourier inversion formula* is defined by

$$f^\vee(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} \, d\xi$$

for each $f \in \mathscr{S}$. The inversion formula establishes the estimate

$$\|f\|_2 = \|\hat{f}\|_2$$

for all $f \in \mathscr{S}$, whence we can extend the Fourier transform operator onto all of L^2 such that the above norm estimate holds for all $f \in L^2$. This is *Plancherel's theorem*, and it tells us that the Fourier transform is a bounded linear operator from L^2 to L^2. The Riesz-Throin interpolation theorem now establishes the *Hausdorff-Young inequality*

$$\|\hat{f}\|_{p'} \leq \|f\|_p$$

for all $1 \leq p \leq 2$, where p' is the *conjugate exponent* of p, defined by the identity $p^{-1} + (p')^{-1} = 1$. This shows that the Fourier transform is a bounded linear operator from L^p to $L^{p'}$, and thus we can make sense of the Fourier transform of L^p functions for $1 \leq p \leq 2$. Unfortunately, the Fourier transform is not a bounded operator into another Lebesgue space for $p > 2$. Therefore, we shall restrict our attention to functions in $L^1 \cap L^p$, so as to guarantee the existence of the Fourier transform.

Having defined the Fourier transform on Lebesgue spaces, we now consider the following question:

Question. *How do we make sense of the equality*

$$f = (\hat{f})^{\vee}?$$

for general $f \in L^p$? More precisely, if

$$S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi,$$

in what sense does $S_R f$ converge to f as $R \to \infty$?

For $n = 1$, there is the pointwise almost-everywhere result, established for L^2 functions by Lennart Carleson and extended to L^p functions for all $1 < p < \infty$ by Richard Hunt two years later:

Theorem 5 (Carleson-Hunt, 1966 & 1968)**.** *Fix $1 < p < \infty$, let $f \in L^p(\mathbb{R})$, and assume that $\hat{f}$ exists. Then*

$$f(x) = \lim_{R \to \infty} S_R f(x)$$

almost everywhere.

$p = 1$ is hopeless, for Andrey Kolmogorov exhibited an L^1 function whose Fourier inversion fails to converge at every point. As for $n > 1$, we must first consider the method of summing the integrals. If the integral is summed over rectangles, then the multivariate almost-everywhere convergence follows from the one-dimensional result. The corresponding result over balls is still open, as of yesterday (and most likely today as well).

How about the L^p convergence? For $n = 1$, we have the classical theorem of Marcel Riesz:

Theorem 6 (M. Riesz, 1928)**.** *Fix $1 < p < \infty$, let $f \in L^p(\mathbb{R})$, and assume that $\hat{f}$ exists. Then*

$$\lim_{R \to \infty} \|f - S_R f\|_p = 0.$$

How about $n > 1$? Again, spherical summation is the difficult part of the multivariate case. The L^2 convergence is a classic result, but do we have L^p convergence for $p \neq 2$? This problem, commonly referred to as the *ball multiplier problem*, is the first instance of the *Kakeya problem* we shall investigate:

Theorem 7 (Fefferman, 1971)**.** *The spherical summation of the Fourier inversion formula converges in the norm topology of L^p if only if $p = 2$ viz.,*

$$\lim_{R \to \infty} \|f - S_R f\|_p = 0.$$

for all $f \in L^p$ whose Fourier transform exist if only if $p = 2$.

For technical reasons[3], if spherical summation works for $n = k$, then it works for $n = k - 1$. Therefore, it is enough to settle the ball multiplier problem for $n = 2$. Likewise, by a "standard duality argument[4]", it is enough to consider the case

[3] by de Leeuw's theorem

[4] by the F. Riesz representation theorem

$p > 2$. Finally, a basic theorem in functional analysis[5] implies that the convergence of spherical summation is equivalent to the norm estimate

$$\|S_R f\|_p \le k_p \|f\|_p,$$

for all sufficiently large R, and a scaling argument shows that this norm estimate is, in fact, equivalent to the single norm estimate

$$\|S_1 f\|_p \le k_p \|f\|_p.$$

It therefore suffices to exhibit, for each $N > 0$ and every $p > 2$, a non-zero function f such that

$$\|S_1 f\|_p \ge N \|f\|_p.$$

Having made the simplifying assumptions, the general idea of Fefferman's proof is not too difficult to understand. Here we quote [Łab]:

> Fefferman does this by letting f be a sum of characteristic functions of long and thin tubes, multiplied by appropriate phase factors. It turns out that S essentially shifts each tube by a fixed distance in the "long" direction. Now suppose that the shifted tubes form something close to a Kakeya set; then the support of Sf is small, and so by Hölder's inequality its L^p norm is large.

We may now ask ourselves just how badly the convergence fails. More precisely, we consider the *Bochner-Riesz mean*

$$S_R^\delta f(x) = \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta d\xi$$

for each $R > 0$ and $\delta \ge 0$. S_R^0 is the regular spherical summation, whose lack of convergence we have just shown. What about $\delta > 0$? Again, the similar reduction argument as in the solution to the ball multiplier problem implies that the L^p convergence of the Bochner-Riesz mean is equivalent to the norm estimate

$$\|S_1^\delta f\|_p \le k_p \|f\|_p.$$

For technical reasons[6], the above estimate does not hold unless

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\delta + 1}{2n},$$

but this does not say anything about when the estimate does hold. Therefore, it is natural to conjecture the following:

Conjecture 8 (Bochner-Riesz conjecture)**.** *If $\delta > 0$, $1 \le p \le \infty$, and*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\delta + 1}{2n},$$

then

$$\lim_{R \to \infty} \|S_R^\delta f - f\|_p = 0$$

for all $f \in L^p$.

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[5]the Banach-Steinhaus theorem, also known as the uniform boundedness principle

[6]by Herz's theorem

4. The Kakeya Conjectures

Let us now return to the question from the beginning of the talk. As it turns out, a modification of Fefferman's construction would disprove the Bochner-Riesz conjecture if we can solve the n-dimensional needle problem on a set "too small to be considered n-dimensional". To state this more precisely, we need the notion of Hausdorff measure and the corresponding Hausdorff dimension.

Definition 9. Fix $m \in \mathbb{N}$, and let ω_m be the m-dimensional Lebesgue measure of the closed unit ball in $\mathbb{R}^m$. The *m-dimensional Hausdorff outer measure* $\mathscr{H}^m$ is defined for every subset $E \subseteq \mathbb{R}^n$ by

$$\mathscr{H}^m(E) = \lim_{\delta \to 0} \inf_{\substack{E \subseteq \bigcup S_j \\ \operatorname{diam}(S_j) \leq \delta}} \sum_{j=1}^{\infty} \omega_m \left(\frac{\operatorname{diam}(S_j)}{2} \right)^m,$$

where the infimum is taken over a countable cover $(S_j)_{n=1}^{\infty}$ of E of diameter at most δ. More generally, we let

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(s/2+1)}$$

for each $s > 0$ and define the *s-dimensional Hausdorff outer measure* $\mathscr{H}^s$ of $E \subseteq \mathbb{R}^n$ to be

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \inf_{\substack{E \subseteq \bigcup S_j \\ \operatorname{diam}(S_j) \leq \delta}} \sum_{j=1}^{\infty} \omega_s \left(\frac{\operatorname{diam}(S_j)}{2} \right)^s.$$

The *Hausdorff dimension* of $E \subseteq \mathbb{R}^n$ is defined to be either the supremum of all s such that $\mathscr{H}^s(E) = \infty$ or the infimum of all s such that $\mathscr{H}^s(E) = 0$.

The Hausdorff measure is defined from the Hausdorff outer measure via the standard Carathéodory construction. Here we have chosen the normalization so that the n-dimensional Hausdorff measure coincides with the n-dimensional Lebesgue measure. Note that $\mathscr{H}^0$ is the counting measure. The one-dimensional Hausdorff measure $\mathscr{H}^1$ of a rectifiable curve is its length. In general, the m-dimensional Hausdorff measure of an m-manifold coincides with its canonical surface measure. The Hausdorff dimension of $E \subseteq \mathbb{R}^n$ specifies the "fair" dimension of E in the following sense: while a curve and a surface both have zero "3-dimensional measure", it would be unfair to say that they are "equally small"—indeed, a curve would have zero "2-dimensional measure" as well.

We are now ready to state the following:

Conjecture 10 (Kakeya Set Conjecture)**.** *Every Kakeya set in $\mathbb{R}^n$ is of Hausdorff dimension n.*

The remark in the beginning of this section, stated precisely, states that the Bochner-Riesz conjecture implies the Kakeya set conjecture.

Even without relying on the as-of-yet-nonexistent proof of the Bochner-Riesz conjecture, we can transform the Kakeya set conjecture into a problem in harmonic analysis. To do this, we let

$$T_e^\delta(a) = \left\{ x \in \mathbb{R}^n : |(x-a)\cdot e| \leq \frac{1}{2} \text{ and } |x - (x \cdot e)e| \leq \delta \right\}$$

for each $\delta > 0$, $e \in \mathbb{S}^{n-1}$, and $a \in \mathbb{R}^n$: this is essentially the δ-neighborhood of the unit line segment in the direction of e centered at a. Using this, we define the *Kakeya maximal function* of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ to be the function

$$f^*_\delta(e) = \sup_{a\in\mathbb{R}^n} \frac{1}{m(T^\delta_e(a))} \int_{T^\delta_e(a)} |f(x)|\, dx$$

for each $e \in \mathbb{S}^{n-1}$. The Kakeya maximal function conjecture concerns a particular L^p estimate of the maximal function in the case $p = n$:

Conjecture 11 (Kakeya Maximal Function Conjecture)**.** *For all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|f^*_\delta\|_{L^n(\mathbb{S}^{n-1})} \le C_\varepsilon \delta^{-\varepsilon} \|f\|_n.$$

Here $\|\cdot\|_{L^n(\mathbb{S}^{n-1})}$ is the norm of the Lebesgue space on $\mathbb{S}^{n-1}$, defined to be

$$\|f\|_{L^n(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} |f(x)|^n\, d\sigma(x).$$

$d\sigma$ is the canonical surface measure on $\mathbb{S}^{n-1}$, or alternatively the $(n-1)$-dimensional Hausdorff measure on $\mathbb{S}^{n-1}$.

As expected, the Kakeya maximal function conjecture implies the Kakeya set conjecture. The formulation of the Kakeya maximal function conjecture is in part inspired by another problem in harmonic analysis: namely, that of restricting the Fourier transform to a lower-dimensional subset of $\mathbb{R}^n$. First, given a function $f : \mathbb{S}^{n-1} \to \mathbb{C}$, we consider the Fourier transform

$$\widehat{f\, d\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} f(\xi) e^{-2\pi i x\cdot\xi}\, d\sigma(x).$$

The restriction problem, stated precisely, is the following:

Conjecture 12 (Stein's Restriction Conjecture)**.** *If $f \in L^\infty(\mathbb{S}^{n-1})$, then, for each $q > \frac{2n}{n-1}$, there exists a constant $C_q > 0$ such that*

$$\|\widehat{f\, d\sigma}\|_q \le C_q \|f\|_\infty.$$

It is known that the Bochner-Riesz conjecture implies the restriction conjecture, and that the restriction conjecture implies the Kakeya maximal function conjecture. Here is the chain of implications:

Theorem 13. *Bochner-Riesz $\Rightarrow$ Restriction $\Rightarrow$ maximal Kakeya $\Rightarrow$ set Kakeya.*

The Kakeya problem is an active area of research, with contributions from some of the finest analysts of our time. See [Łab] or [Tao01] for a quick survey. More systematic treatments are given in, for example, [Wol03] or [Tao]

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