WHAT IS THE STRICHARTZ ESTIMATE?

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Given the wave equation

$$\Box u(t,x) = F(t,x)$$

in $\mathbb{R}^{1+3}$, where the *D'Alembertian* $\Box$ is defined to be

$$\Box = \partial_t^2 - \Delta,$$

with intial data

$$\begin{aligned} u(0,x) &= f(x); \\ \partial_t u(0,x) &= g(x), \end{aligned}$$

the *Strichartz estimate* for u is

$$\|u\|_{L^4(\mathbb{R}^{1+3})} \leq c(\|f\|_{H^{1/2}(\mathbb{R}^3)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} + \|F\|_{L^{4/3}(\mathbb{R}^{1+3})}).$$

Here $L^p(\mathbb{R}^{1+3})$ is the Lebesgue space of order p on $\mathbb{R}^4$, and $H^p(\mathbb{R}^3)$ is the Sobolev space of order p on $\mathbb{R}^3$. We shall define the Sobolev space in due course.

For $f \in L^1(\mathbb{R}^n)$, the *Fourier transform* of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}.$$

If f is C^1 and goes to zero at ∞, then

$$\begin{aligned} \widehat{\partial_j f}(\xi) &= \int_{\mathbb{R}^n} (\partial_j f(x))\, e^{-2\pi i \xi \cdot x}\, dx \\ &= -\int_{\mathbb{R}^n} f(x) \left(\partial_j e^{-2\pi i \xi \cdot x}\right) dx \\ &= -\int_{\mathbb{R}^n} (-2\pi i \xi_j) f(x) e^{-2\pi i \xi \cdot x}\, dx \\ &= (2\pi i \xi_j) \hat{f}(\xi). \end{aligned}$$

Taken as a bounded operator on $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, the Fourier transoform can be extended onto $L^2(\mathbb{R}^n)$. The standard result in the L^2-theory of the Fourier transform is *Plancherel's theorem*, which states that $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ for all $f \in L^2(\mathbb{R}^n)$. An immediate consequence is that

$$\|\partial_j f\|_{L^2} = \|\widehat{\partial_j f}\|_{L^2} = \|2\pi i \xi_j \hat{f}\|_{L^2}.$$

Taking a cue from the above calculation, we define, for each $s \geq 0$, the space $H^s(\mathbb{R}^n)$ of L^2 funcitons with s derivatives by

$$\dot{H}^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \||\xi|^s \hat{f}(\xi)\|_{L^2} < \infty\}.$$

Note that there are two natural norms for a function $f \in \dot{H}^s(\mathbb{R}^n)$: namely, $\|f\|_{L^2} + \|\partial_j f\|_{L^2}$ and $\|\partial_j f\|_{L^2}$. The first norm, however, is not homogeneous, and so we take the second norm to be the norm of the space $\dot{H}^s(\mathbb{R}^n)$. Indeed, the dot in $\dot{H}^s(\mathbb{R}^n)$ refers to the homogeneity of the space. Henceforth, we shall write

$$\|f\|_{H^s} = \|f\|_{H^s(\mathbb{R}^n)} = \||\xi|^s \hat{f}(\xi)\|_{L^2}.$$

A particularly useful space of functions in Fourier analysis is the *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$, which is the space of $\mathscr{C}^\infty$-functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)|\, dx < \infty$$

for each integer N and every multi-index $\alpha \in \mathbb{Z}^n$. We can now state the L^2 *restriction theorem of the fourier transform in* $\mathbb{R}^4$ *for the light cone*, which states that there is a uniform constant c such that for each $G \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\left(\int_{\mathbb{R}^3} |\hat{G}(|\xi|,\xi)|^2 \frac{d\xi}{|\xi|}\right)^{1/2} \le c\|G\|_{L^{4/3}(\mathbb{R}^{1+3})}.$$

We shall see that the restriction theorem is intimitely connected to the Strichartz estimate.

Let us now consider the solution of the wave equation given by the Fourier transform. Since $\partial_t^2 u(t,x) - \Delta u(t,x) = 0$, we have

$$\widehat{\partial_t^2 u}(t,\xi) - \widehat{\Delta u}(t,\xi) = 0.$$

Noting that each $h \in \mathscr{C}^2$ satisfies the identity

$$\widehat{\partial_{x_i}^2 h} = (-i\xi)^2 \hat{h}(\xi),$$

we now have

$$\widehat{\partial_t^2 u}(t,\xi) - (-i\xi)^2 \hat{u}(t,\xi) = 0,$$

which is an ordinary differential equation in t, for a fixed ξ. In fact, the solutions can expressed in the form

$$\hat{u}(t,\xi) = \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|} + \hat{f}(\xi)\cos t|\xi|,$$

where f and g are the initial conditions of the wave eqaution.

We now prove that the Strichartz estimate implies the L^2 restriction. We define an operator T by

$$Tf(t,x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi + it|\xi|} \frac{\hat{f}(\xi)}{|\xi|^{1/2}}\, d\xi = \int e^{ix\cdot\xi+it|\xi|}\hat{h}(\xi)\, d\xi.$$

where $\hat{h} = \frac{\hat{f}}{\xi^{1/2}}$

Notice, this is the solution to the wave equation with particular initial data f and g and the right hand side $F \equiv 0$. For,

$$\widehat{Tf} = e^{it|\xi|}\hat{h} = cos(t|\xi|)\hat{h} + i\frac{sin(t|\xi|)}{|\xi|}|\xi|\hat{h}$$

and so just take $f = h$ and g such that $\hat{g} = |\xi|\hat{h}$.

We now claim that T is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^4(\mathbb{R}^{1+3})$ and satisfies the norm estimate

$$\|Tf\|_{L^4(\mathbb{R}^{1+3})} \le c\|h\|_{L^2}.$$

Proof: Because of Strichartz estimate

$$\|Tf\|_{L^{4/3}(\mathbb{R}^4)} \leq c(\|h\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}})$$

From definition, $\|h\|_{\dot{H}^{1/2}} = \|\xi^{1/2}\hat{h}\| = \|\xi^{1/2}\frac{\hat{f}}{\xi^{1/2}}\| = \|\hat{f}\| = \|f\|$ where all the norms on the right are in L^2 and by Plancherel in the last equality.

Also, $\|g\|_{\dot{H}^{-1/2}} = \|\xi^{-1/2}\hat{g}\| = \|\xi^{-1/2}|\xi|\hat{h}\| = \|\hat{f}\| = \|f\|$ by once again using definition of the Sobolove norm and the defition of h.

So claim proved.

Now, let us compute the dual of T, call it $T^* : (L^4(\mathbb{R}^4))^* \to (L^2(\mathbb{R}^3))^*$, that is, $T^* : (L^{4/3}(\mathbb{R}^4)) \to (L^2(\mathbb{R}^3))$

Take some $a \in L^2\mathbb{R}^3$. Fix $f \in L^{4/3}(\mathbb{R})^4$.

In the following, if we use $\tilde{\alpha}$ it means the space-time Fourier Transform as opposed to just the usual spatial one $\alpha^\wedge$

Then,

$$\begin{aligned}
< T^*f, a > &= < f, Ta > \\
\int_{\mathbb{R}^3} (T^*f)\bar{a} &= \int_{\mathbb{R}^4} (f(t,x))\overline{(Ta)(t,x)} \\
&= \int_{\mathbb{R}^4} (f(t,x))\overline{\int_{\mathbb{R}^3} e^{ix\cdot\xi+it|\xi|}\frac{\hat{a}(\xi)}{|\xi|^{1/2}}} \\
&= \int_{\mathbb{R}^4} (f(t,x))\int_{\mathbb{R}^3} e^{-ix\cdot\xi-it|\xi|}\frac{\bar{a}^\vee(\xi)}{|\xi|^{1/2}} \\
&= \int_{t\in\mathbb{R}^1}\int_{x\in\mathbb{R}^3}\int_{\xi\in\mathbb{R}^3} (f(t,x))e^{-ix\cdot\xi-it|\xi|}\frac{\bar{a}^\vee(\xi)}{|\xi|^{1/2}} \\
&= \int_{t\in\mathbb{R}^1}\int_{\xi\in\mathbb{R}^3}\int_{x\in\mathbb{R}^3} (f(t,x))e^{-ix\cdot\xi-it|\xi|}\frac{\bar{a}^\vee(\xi)}{|\xi|^{1/2}} \\
&= \int_{\xi\in\mathbb{R}^3}\int_{t\in\mathbb{R}^1}\int_{x\in\mathbb{R}^3} (f(t,x))e^{-ix\cdot\xi-it|\xi|}\frac{\bar{a}^\vee(\xi)}{|\xi|^{1/2}} \\
&= \int_{\xi\in\mathbb{R}^3}\frac{\bar{a}^\vee(\xi)}{|\xi|^{1/2}}(\tilde{f}(|\xi|,\xi)) \\
&= \int_{x\in\mathbb{R}^3}\bar{a}(x)(\frac{\tilde{f}(|\xi|,\xi)}{|\xi|^{1/2}})^\wedge(x)
\end{aligned}$$

This means, that T^*f is precisely $(\frac{\tilde{f}(|\xi|,\xi)}{|\xi|^{1/2}})^\wedge(x)$

So the fact that it is a bounded linear operator from the aforementioned spaces means that $\|(\frac{\tilde{f}(|\xi|,\xi)}{|\xi|^{1/2}})^\wedge\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^{4/3}(\mathbb{R}^3)}$

Now using Plancherel on the left side this reads:

$\|(\frac{\tilde{f}(|\xi|,\xi)}{|\xi|^{1/2}})\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^{4/3}(\mathbb{R}^3)}$ which is precisely what we wanted - that Strichartz estimate implies that Fourier transform of functions in $\mathbb{R}^4$ restricts continously on to the lightcone in $\mathbb{R}^4$