Math 111 Exam 1 Solutions

Problem 1. Simplify the following expressions. Express the final answers with positive exponents.

(a) $\left(\frac{3x^2}{5yz^3}\right)^{-2}\left(\frac{xy^2}{5z}\right)$ (5 points.)

Solution.

$$\begin{aligned}
\left(\frac{3x^2}{5yz^3}\right)^{-2}\left(\frac{xy^2}{5z}\right) &= (3^1 5^{-1} x^2 y^{-1} z^{-3})^{-2}(5^{-1}xy^2z^{-1}) \\
&= (3^{-2}5^2x^{-4}y^2z^6)(5^{-1}xy^2z^{-1}) \\
&= 3^{-2}5^{2-1}x^{-4+1}y^{2+2}z^{6-1} \\
&= 3^{-2}5^1x^{-3}y^4z^5 \\
&= \frac{5y^4z^5}{9x^{-3}}
\end{aligned}$$

(b) $(3y - x^2)^2$ (5 points.) *Solution.*

$$\begin{aligned}
(3y - x^2)^2 &= 9y^2 - 2 \cdot 3y \cdot x^2 + x^4 \\
&= 9y^2 - 6yx^2 + x^4
\end{aligned}$$

Problem 2. Express the following in simplest **radical** form.

(a) $\sqrt[3]{\frac{54x^7}{2y^6z^4}}$ (5 points.)

Solution.

$$\begin{aligned}
\sqrt[3]{\frac{54x^7}{2y^6z^4}} &= \left(\frac{54x^7}{2y^6z^4}\right)^{1/3} \\
&= (27x^7y^{-6}z^{-4})^{1/3} \\
&= 27^{1/3}x^{7/3}y^{-6/3}z^{4/3} \\
&= 3x^{7/3}y^{-2}z^{4/3} \\
&= \frac{3x^{7/3}z^{4/3}}{y^2} \\
&= \frac{3\sqrt[3]{x^7z^4}}{y^2} \\
&= \frac{3x^2z\sqrt[3]{xz}}{y^2}
\end{aligned}$$

(b) $\left(\sqrt[3]{5} - 1\right)\left(\sqrt[3]{5^2} + \sqrt[3]{5} + 1\right)$ (5 points.)

Solution. Recall the difference-of-two-cubes formula

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2).$$

We thus see that

$$\begin{aligned}\left(\sqrt[3]{5}-1\right)\left(\sqrt[3]{5^2}+\sqrt[3]{5}+1\right) &= (\sqrt[3]{5})^3-1^3\\ &= 5-1\\ &= 4\end{aligned}$$

Problem 3. Use the method of **completing the square** to solve the following quadratic equation (10 points.)

$$2x^2 = 5x+3$$

Solution.

$$\begin{aligned}2x^2 &= 5x+3\\ 2x^2-5x &= 3\\ x^2-\frac{5}{2}x &= \frac{3}{2}\\ x^2-\frac{5}{2}x+\frac{25}{4} &= \frac{3}{2}+\frac{25}{4}\\ \left(x-\frac{5}{4}\right)^2 &= \frac{31}{4}\\ x-\frac{5}{4} &= \pm\sqrt{\frac{31}{4}}\\ x &= \frac{5}{4}\pm\sqrt{\frac{31}{4}}\\ x &= \frac{5}{4}\pm\frac{\sqrt{31}}{2}\end{aligned}$$

Problem 4. Solve each of the following equations for x.

(a) $\sqrt{5-x}=x-3$ (10 points.)

Solution.

$$\begin{aligned}\sqrt{5-x} &= x-3\\ 5-x &= (x-3)^2\\ 5-x &= x^2-6x+9\\ 0 &= x^2-5x+4\\ 0 &= (x-1)(x-4)\\ x &= 1,4\end{aligned}$$

Only $x=4$ works (check it!)

(b) $\dfrac{2xy-3}{x-3y}=7y^2$

Solution.

$$\begin{aligned}
\frac{2xy-3}{x-3y} &= 7y^2 \\
\frac{2xy-3}{x-3y} &= \frac{7y^2(2xy-3)}{x-3y} \\
\frac{2xy-3}{x-3y} &= \frac{14xy^3-21y^2}{x-3y} \\
\frac{2xy-3-14xy^3+21y^2}{x-3y} &= 0 \\
\frac{2x(y-7y^3)-(3-21y^2)}{x-3y} &= 0 \\
2x(y-7y^3)-(3-21y^2) &= 0 \\
2x(y-7y^3) &= 3-21y^2 \\
2x &= \frac{3-21y^2}{y-7y^3} \\
2x &= \frac{3(1-7y^2)}{y(1-7y^2)} \\
2x &= \frac{3}{y} \\
x &= \frac{3}{2y}
\end{aligned}$$

Problem 5. Simplify the following expression. (10 points.)

$$\frac{2}{x-1} - \frac{1}{x^2-1} + 1$$

Solution.

$$\begin{aligned}
\frac{2}{x-1} - \frac{1}{x^2-1} + 1 &= \frac{2}{x-1} - \frac{1}{(x+1)(x-1)} + 1 \\
&= \frac{2(x+1)}{(x+1)(x-1)} - \frac{1}{(x+1)(x-1)} + \frac{(x+1)(x-1)}{(x+1)(x-1)} \\
&= \frac{2x+1}{(x+1)(x-1)} - \frac{1}{(x+1)(x-1)} + \frac{x^2-1}{(x+1)(x-1)} \\
&= \frac{2x+1-1+x^2-1}{(x+1)(x-1)} \\
&= \frac{x^2+2x-1}{(x+1)(x-1)}
\end{aligned}$$

Problem 6. Mary drove from Amity to Belleville at a speed of 50 mi/h. On the way back, she drove at 60 mi/h. The total trip took $4\frac{2}{5}$ hours of driving time. **Find the distance between the two cities**. (10 points.)

Solution. Let t be the time, in hours, it took Mary to drive from Amity to Belleville. It immediately follows that $4\frac{2}{5} - t$ is the time, in hours, it took her to drive from Belleville to Amity. Thus,

$$50t = 60\left(4\frac{2}{5} - t\right)$$

We proceed to solve this.

$$\begin{aligned} 50t &= 60\left(4\frac{2}{5} - t\right) \\ 50t &= 60\left(\frac{22}{5} - t\right) \\ 50t &= 264 - 60t \\ 110t &= 264 \\ t &= \frac{264}{110} \\ t &= \frac{12}{5} \end{aligned}$$

Now, use the distance formula

$$(\text{distance}) = (\text{speed}) \times (\text{time})$$

to compute the distance. Indeed,

$$50 \times \frac{12}{5} = 120 \text{ miles}$$

Problem 7. Solve the following inequalities and leave your answer in both interval and graph forms.

(a) $\left|\frac{2x}{3} - 4\right| \geq 2$ (8 points.)

Solution.

$$\left|\frac{2x}{3} - 4\right| \geq 2$$

implies

$$\frac{2x}{3} - 4 \geq 2 \text{ or } \frac{2x}{3} - 4 \leq -2.$$

We solve the first inequality:

$$\begin{aligned} \frac{2x}{3} - 4 &\geq 2 \\ \frac{2x}{3} &\geq 6 \\ 2x &\geq 18 \\ x &\geq 9 \end{aligned}$$

...and the second inequality:

$$\begin{aligned} \frac{2x}{3} - 4 &\leq -2 \\ \frac{2x}{3} &\leq 2 \\ 2x &\leq 6 \\ x &\leq 3 \end{aligned}$$

We conclude that

$$x \geq 9 \text{ or } x \leq 3.$$

(b) $\frac{2x}{2x-3} \geq 2$ (10 points.)
Solution.

$$\begin{aligned} \frac{2x}{2x-3} &\geq 2 \\ \frac{2x}{2x-3} &\geq \frac{2(2x-3)}{2x-3} \\ \frac{2x}{2x-3} &\geq \frac{4x-6}{2x-3} \\ 0 &\geq \frac{4x-6}{2x-3} - \frac{2x}{2x-3} \\ 0 &\geq \frac{4x-6-2x}{2x-3} \\ 0 &\geq \frac{2x-6}{2x-3} \end{aligned}$$

The critical points are $x = 3$ and $x = \frac{3}{2}$. We make a table:

	$x < \frac{3}{2}$	$\frac{3}{2} < x < 3$	$x > 3$
$2x-6$	$-$	$-$	$+$
$2x-3$	$-$	$+$	$+$
$\frac{2x-6}{2x-3}$	$+$	$-$	$+$

We conclude that

$$\frac{3}{2} \leq x \leq 3.$$

Problem 8. (a) If $A = (2, 4)$ and $M = (1/2, 6)$, find the coordinates of the point $B = (x, y)$ such that M is the midpoint of the line segment AB (4 points.)

Solution. Let $B = (x, y)$. Then

$$M = \left(\frac{2+x}{2}, \frac{4+y}{2}\right) = \left(\frac{1}{2}, 6\right)$$

Clearly, $x = -1$ and $y = 8$, whence it follows that $B = (-1, 8)$.

(b) Find an equation of the circle for which the line segment AB is a diameter. (6 points.)
Solution. First, the length of AB is:

$$d = \sqrt{(-1-2)^2 + (8-4)^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

Hence, the circle we want has radius $\frac{5}{2}$, and is centered at $M = (1/2, 6)$. The equation is:

$$\left(x - \frac{1}{2}\right)^2 + (y-6)^2 = \left(\frac{5}{2}\right)^2$$

(c) Find the center and the radius of the circle given by the equation (6 points.)

$$x^2 + y^2 - 6x + 10y + 25 = 0$$

Solution.

$$\begin{aligned} x^2 + y^2 - 6x + 10y + 25 &= 0 \\ x^2 - 6x + y^2 + 10y + 25 &= 0 \\ (x^2 - 6x) + (y^2 + 10y + 25) &= 0 \\ (x^2 - 6x) + (y+5)^2 &= 0 \\ (x^2 - 6x + 9) + (y+5)^2 &= 9 \\ (x-3)^2 + (y+5)^2 &= 3^2 \end{aligned}$$

whence it follows that $C = (3, -5)$ and $R = 3$.

Problem 9. (Extra credit) Simplify the following expression. (5 points)

$$1 + \cfrac{1}{1 + \cfrac{1}{1+x}}$$

Solution

$$\begin{aligned} 1 + \cfrac{1}{1 + \cfrac{1}{1+x}} &= 1 + \cfrac{1}{\left(\cfrac{1+x}{1+x} + \cfrac{1}{x}\right)} \\ &= 1 + \cfrac{1}{\left(\cfrac{2+x}{1+x}\right)} \\ &= 1 + 1 \div \frac{2+x}{1+x} \\ &= 1 + 1 \times \frac{1+x}{2+x} \\ &= 1 + \frac{1+x}{2+x} \\ &= \frac{2+x}{2+x} + \frac{1+x}{2+x} \\ &= \frac{2x+3}{2+x} \end{aligned}$$