POLYNOMIALS

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Abstract. This packet is designed to supplement the lectures on polynomials, as presented in Math 111 or Math 115 at Rutgers University. As such, it is not meant to be used as a first exposure to the material: the explanation for computational methods is minimal, and typical results from the chapter may be quoted without further elaboration.

1. Polynomial Functions

We assume that the reader is acquainted with the definition of a polynomial. To check, we begin with a simple exercise.

1.1. **Exercise.** Which of the following is a polynomial?

(1) 5
(2) $x^5 + 8x^{-2}$
(3) $x^{7/2} - 9x$
(4) $x/2 + 8x$

A polynomial function is a function that evaluates a polynomial; viz.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

1.2. **Exercise.** Let f and g be polynomial functions. Which of the following is *not* a polynomial?

(1) $(f + g)(x)$
(2) $(f - g)(x)$
(3) $(f \cdot g)(x)$
(4) $(f/g)(x)$.

As a function, a polynomial function has a graph associated with it. To graph a polynomial function, we need four facts about the function:

- the behavior of the function as x approaches ∞
- the behavior of the function as x approaches $-\infty$
- the zeroes of the function
- the multiplicities of the zeroes of the function

Intuitively, if we plug a very big number x into a polynomial function $f(x)$, we get a very big number. From this observation, four possible conclusions follow:

It remains to devise a method for determining the behavior of $f(x)$ for very large x. Recall that a polynomial is a finite sum of monomial. We can think of a polynomial as a group of politically active monomials, struggling to have their opinions count more than those of the others. Here is a simple example: take $f(x) = x + 1$. This is the sum of $f_1(x) = x$ and $f_2(x) = 1$. Both f_1 and f_2, while residing together in f, struggle to make f look more like themselves. If you look at the values of x near 0, it seems as though f_2 is winning the battle: $f(0.1) = 0.9$ is closer to $f_2(0.1) = 1$

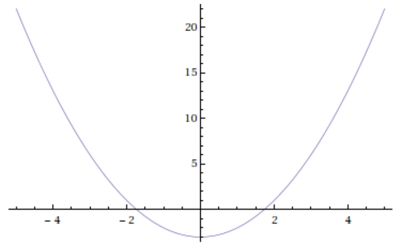

FIGURE 1. If x is a big positive number, then $f(x)$ is a big positive number; if x is a big negative number, then $f(x)$ is a big positive number.

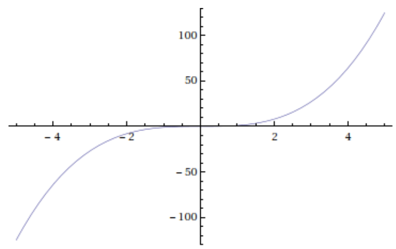

FIGURE 2. If x is a big positive number, then $f(x)$ is a big positive number; if x is a big negative number, then $f(x)$ is a big negative number.

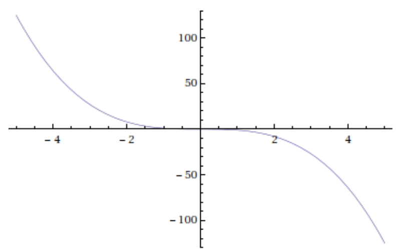

FIGURE 3. If x is a big positive number, then $f(x)$ is a big negative number; if x is a big negative number, then $f(x)$ is a big positive number.

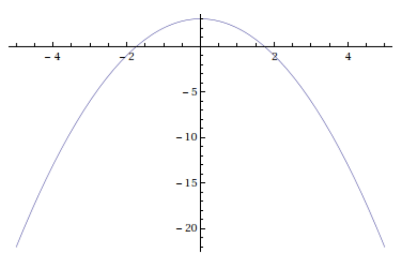

FIGURE 4. If x is a big positive number, then $f(x)$ is a big negative number; if x is a big negative number, then $f(x)$ is a big negative number.

than $f_1(0.1) = 0.1$. If you take a sufficiently large value of x, however, it becomes evident that f looks like f_1, not f_2: $f(10000000) = 10000001$ is clearly closer to $f_1(10000000) = 10000000$ than $f_2(10000000) = 1$. Since the behaviors of a polynomial function at very large values of x determine the overall shape of the graph, we can therefore conclude that the graph of f looks like the graph of f_1; in fact, f is a translation of f_1.

Let us generalize this argument. For an arbitrary polynomial function $f(x)$, the monic term with the biggest exponent eventually dominate the behavior of the function, as x gets bigger. This monic term has the majority in the house; no matter how many of the remaining terms collaborate, they still won't be able to overpower this one term. In fact, letting $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, we can conclude that the graph of f looks like that of $f_n(x) = a_n x^n$.

It is, then, sufficient to analyze one specific monic term to determine the behavior of a polynomial function at very large values of x. The analysis is done again in four cases:

Case 1. $a_n > 0$, and n is even. Since n is even, x^n is positive for all x except $x = 0$. $a_n > 0$, hence $a_n x^n > 0$. Therefore, if x is a big positive number, then $a_n x^n$ is a big positive number. Furthermore, if x is a big negative number, then $a_n x^n$ is a big positive number.

Case 2. $a_n > 0$, and n is odd. Since n is odd, x^n is positive for all $x > 0$, and negative for all $x < 0$. Therefore, if x is a big positive number, then $a_n x^n$ is a big positive number, whereas if x is a big negative number, then $a_n x^n$ is a big negative number.

Case 3. $a_n < 0$, and n is odd. Since n is odd, x^n is negative for all $x > 0$, and positive for all $x < 0$. Therefore, if x is a big positive number, then $a_n x^n$ is a big negative number, whereas if x is a big negative number, then $a_n x^n$ is a big positive number.

Case 4. $a_n < 0$, and n is even. Since n is even, x^n is positive for all x except $x = 0$. $a_n < 0$, hence $a_n x^n < 0$. Therefore, if x is a big positive number, then $a_n x^n$ is a big negative number. Furthermore, if x is a big negative number, then $a_n x^n$ is a big negative number.

1.3. **Exercise.** Determine the behavior of the following polynomial functions, and match each of them with one of the four general shapes presented in pages 1 and 2.

(1) $f(x) = x^5 - 7x^2 + x$
(2) $g(x) = -x^2 - 10x^9 + x^3$
(3) $p(x) = -100000x^{10} + 20x^{20}$
(4) $q(x) = 5x^{44} + 8123423x^{71} - x^{90}$

We defer the discussion of the zeroes of polynomial functions and their multiplicities to section 4.

2. Dividing Polynomials

Recall the process of division in arithmetic. What is, for example, $5 \div 3$? By the process of **long division**, we can easily conclude that the answer is "1 remainder 2." In other words,

$$5 \div 3 = 1 + \frac{2}{3},$$

which we can easily check, applying the definition $5 \div 3 = \frac{5}{3}$. In fact, division always adheres to the following format:

$$\text{dividend} \div \text{divisor} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

We generalize the notion by allowing the dividend and the divisor to be polynomials. For example,

$$(x-5) \div x = 1 - \frac{5}{x}$$

makes sense, for

$$(x-5) \div x = \frac{x-5}{x} = \frac{x}{x} - \frac{5}{x} = 1 - \frac{5}{x}.$$

In order to have a unique outcome for each division, we must impose a restriction on the remainder. Otherwise,

$$(x-5) \div x = 2 - \frac{x+5}{x}$$

is also a correct answer to the previous division problem. We observe that the remainder is a nonnegative number smaller than the divisor when the dividend and the divisor are integers. That is, if we divide the remainder by the divisor, we no longer obtain an integer. Therefore, when performing a polynomial division, we would like the remainder to be sufficiently small that dividing the remainder by the divisor no longer results in a polynomial. This condition is achieved when the degree of the remainder is smaller than the degree of the divisor. Now, then,

$$(x-5) \div x = 1 - \frac{5}{x}$$

is an acceptable answer, whereas

$$(x-5) \div x = 2 - \frac{x+5}{x}$$

is not.

Now, recall the **long division** method. For example:

$$\begin{array}{r} 12 \\ 11\overline{)135} \\ \underline{110} \\ 25 \\ \underline{22} \\ 3 \end{array}$$

We note that $135 \div 11$ is equivalent to $(1 \cdot 10^2 + 3 \cdot 10^1 + 5) \div (1 \cdot 10 + 1)$. The latter looks very similar to polynomial division; in fact,

$$\begin{array}{rr} & x+2 \\ x+1\overline{)} & x^2+3x+5 \\ & \underline{-x^2 \quad -x} \\ & 2x+5 \\ & \underline{-2x-2} \\ & 3 \end{array}$$

As noted, the long division we know and the polynomial long divison are very similar. The similarity, however, is in their methods of computation, and the reader must not assume that polynomial long division always produces an analgous answer. Here is a counterexample:

$$\begin{array}{r} 10 \\ 257\,\overline{)\,2582} \\ \underline{2570} \\ 12 \end{array} \qquad \begin{array}{rrrrrr} & x^2 & & & -\tfrac{7}{2} \\ \cline{2-6} 2x^2+5x+7\,\big) & 2x^4+5x^3 & & +8x & +2 \\ & -2x^4-5x^3-7x^2 & & & \\ \cline{2-3} & & -7x^2 & +8x & +2 \\ & & 7x^2+ & \tfrac{35}{2}x+ & \tfrac{49}{2} \\ \cline{3-5} & & & \tfrac{51}{2}x+ & \tfrac{53}{2} \end{array}$$

2.1. **Exercise.** Compute $(2x^4 + 5x^3 + 8x + 2) \div (2x^2 + 5x + 7)$ again with the added restriction that the coefficients of the quotients be nonnegative integers. Compare the answer with the answer for $2582 \div 257$. Describe the similarities between long division and polynomial long division.

3. Factoring the Polynomials

Theorem 1 (Remainder Theorem)**.** *The remainder of a polynomial $f(x)$ divided by a linear divisor $x - a$ is $f(a)$.*

That is to say, if we wanted to compute the remainder of $(x^5 + x^4 + x^3 + x^2 + x^1 + x + 1) \div (x - 1)$, it suffices to compute $(1)^5 + (1)^4 + (1)^3 + (1)^2 + (1)^1 + 1$.

Proof of the Remainder Theorem. Let $f(x)$ be a polynomial dividend, $(x - a)$ a divisor, $q(x)$ the quotient, and $r(x)$ the remainder. Since $(x - a)$ is of degree 1, $r(x)$ must be a constant. We can thus let $r(x) = R$. Then, we have

$$f(x) = q(x)(x - a) + R.$$

Substituting $x = a$ gives $f(a) = q(a)(a - a) + R = R$. Therefore, $f(a) = R$. □

An immediate consequence of is the factor theorem.

Theorem 2 (Factor Theorem)**.** *A polynimial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.*

Proof of the Factor Theorem. If $(x - a)$ is a factor of $f(x)$, then $f(x) = (x - a)q(x)$ for some $q(x)$. Clearly, $f(a) = (a - a)q(a) = 0$. Conversely, if $f(a) = 0$, then $f(a) = q(a)(a - a) + R = R = 0$, given the quotient $q(x)$ and the divisor $x - a$. $R = 0$, thus $(x - a)$ divides $f(x)$ evenly. We conclude that $x - a$ is a factor of $f(x)$. □

The factor theorem, as the name suggests, is used to factor polynomials. As an example, let us factor $f(x) = x^3 - 6x^2 + 11x - 6$. $f(1) = 1 - 6 + 11 - 6 = 0$, hence $(x - 1)$ is a factor of $f(x)$. By polynomial division, we have $f(x) = (x - 1)(x^2 - 5x + 6)$. Set $g(x) = x^2 - 5x + 6$. $g(2) = 1 - 5 + 6 = 0$, hence $(x - 2)$ is a factor of $g(x)$, thus of $f(x)$. Again by polynomial division, we have $f(x) = (x - 1)(x - 2)(x - 3)$, and we are done.

Using the factor theorem involves guesswork, as we must plug in a number to a polynomial function to test whether the linear divisor is a factor. The following theorem imposes a constraint on rational roots of polynomial equations, making the procedure swifter:

Theorem 3 (Rational Root Theorem)**.** *Given a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$ with integer coefficients, each rational solution of the equation is of the form $x = \pm\dfrac{p}{q}$, where p is an integer factor of the constant term a_0 and q an integer factor of the leading coefficient a_n.*

The application of the theorem is as thus: If r is a root of the polynomial $f(x)$, then $f(x)$ must include $(x - r)$ as its factor; we therefore produce a list of possible rational roots, and check each candidate via synthetic division. In the previous example, the possible rational roots of $x^3 - 6x^2 + 11x - 6 = 0$ are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. Of these, only 1, 2, and 3 are the roots of the equation.

3.1. **Exercise.** In view of the Rational Root Theorem, produce the list of all possible rational roots of each of the following polynomial. Then, apply the Factor Theorem to factor the polynomial completely. Clearly indicate where each theorem is used.

(1) $f(x) = x^3 - 4x^2 - 11x + 30$
(2) $g(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$
(3) $h(x) = x^3 - x^2 - 4x - 6$

4. Graphing the Polynomials

With the newly developed tools, we can now graph polynomial functions with much greater accuracy. Let

$$f(x) = (x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) q(x)$$

be a completely factored polynomial function. Each r_k is real, and $q(x)$ is a prime polynomial, indivisible by any real linear divisor.

4.1. **Exercise.** Convince yourself that $f(x) = 0$ at $x = r_1, r_2, \ldots, r_n$ and nowhere else. Why is it the case?

Recall that a point x_0 is an x-intercept of the function f if $f(x_0) = 0$. Hence, $\{r_1, r_2, \ldots, r_n\}$ is the complete set of x-intercepts of f. At each x-intercept, the graph of f can behave in either of the following ways: it could (1) go through the x-axis, or (2) touch the x-axis and bounce back.

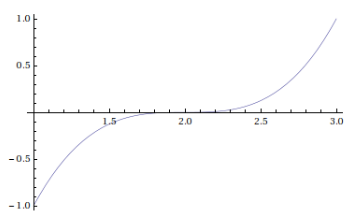

Figure 5. The graph of $f(x) = (x - 2)^3$ goes through the x-axis at $x = 2$.

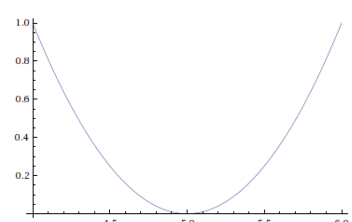

Figure 6. The graph of $f(x) = (x - 5)^2$ bounces at $x = 5$.

To determine the behavior of the function at the x-intercept, we need to introduce the concept of **multiplicity**. Note that we did not force the linear factors to be distinct. In fact, it may even be true that $r_1 = r_2 = r_3 = \cdots = r_n$. We may rewrite the function as

$$f(x) = (x - R_1)^{p_1}(x - R_2)^{p_2} \cdots (x - R_m)^{p_m} q(x),$$

where $R_1, R_2, \ldots, R_m$ are distinct. The multiplicity of a root R_k of a function $f(x)$ is the exponent p_k. That is, the multiplicity of a root measures how many times the root appears in the completely factored form of a polynomial function.

Now, if the multiplicity of the root r is even, then the factor $(x - r)^p$ does not change its sign whether $x > r$ or $x < r$; hence, the graph bounces at $x = r$. On the contrary, if the multiplicity of the root r is odd, then the factor $(x - r)^p$ has a different sign on the left of $x = r$ and on the right of $x = r$; hence, the graph goes through the x-axis at $x = r$. See Figure 5 and 6 for examples.

4.2. **Exercise.** Graph the polynomial functions in Exercise 3.1.

5. The Fundamental Theorem of Algebra

We take for granted the notion of complex numbers in this section. The Fundamental Theorem of Algebra, first proved by Karl Frederich Gauss in 1799, is as follows:

Theorem 4. *Every polynomial of degree 1 or higher with complex coefficients has at least one root.*

Therefore, it makes sense to *attempt* to factor polynomials, for there is always at least one linear factor. An immediate consequence of the theorem is the Linear Factorization Theorem:

Theorem 5. *Every polynomial of degree n has n roots, allowing root multiplicities.*

5.1. **Exercise.** Explain why Theorem 5 is a consequence of Theorem 4.