Functions

What is a function? We have learned that a function f is a *rule* that "assigns to each element x in a set A exactly one element, called $f(x)$, in a set B." Two properties should be clear:

1. That there is *only one* element (of B) assigned to each element of A.

2. That *every element* of A has an element of B assigned to it.

We say that f *maps* A *into* B. A is called the **domain** of the function f, and B the **codomain**.

Given a function f that maps A into B, it may be helpful to think of A as a group of archers with *one* arrow. Each archor's job is to shoot the arrow into B, and read off the label of the target in B the arrow hit. We, of course, assume that archors always succeed in hitting *something* in B, and that one arrow cannot hit two or more targets in B.

Let us make a few observations. First, notice that each archor is responsible for the destination of his arrow; it would be absurd to claim that targets, as if they were sentient beings, determine which arrows will hit them. It is, then, reasonable to say that the act of arrows hitting targets is *dependent* upon the archors, who act *independent* of the targets. We thus arrive at the following definition:

Definition. The symbol that represents an arbitrary element in the domain of a function is called an **independent variable**. In our example, we could let a to represent any element in A and say that a is the independent variable. The symbol that represents any possible value of the function is called a **dependent variable**. If we write $b = f(a)$, then a is the independent variable, and b the dependent variable.

Notice also that there could be some targets in B with no arrows in them. Therefore, it is worth making a distinction between "the collection of all targets" and "the collection of all targets with arrows":

Definition. The **range** of a function f is the set of all possible values of $f(x)$ as the independent variable x varies throughout the domain. The range of f is always a subset of the codomain of f.

Now, we can reformulate the definition of dependent variables in terms of the range of a function:

Definition. The symbol that represents an arbitrary element in the range of a function is called a **dependent variable**.

With the definition of range in mind, we set two conventions. First, we simply forget about the "bad values" in the domain by collapsing the domain. To illustrate this, consider the function $f(x) = \frac{1}{x}$ from the set of all real numberes to the set of all real numbers. Since $f(x)$ is undefined at $x = 0$, we would have to say that $f(x)$ is *not* a function if we were to adhere strictly to the definition of function. Instead, we simple say that the domain of f is *the set of all real numbers except* 0.

Furthermore, we set the range to equal the codomain by throwing away all "unmarked" targets in the codomain. Thus we can disregard the notion of codomain and think of functions as assignment rules from the domain to the range. This will greatly simplify the notion of inverse functions, which we shall now discuss.

Let us recall the archor-target analogy once more. In some cases, it is possible for each target to be hit by exactly one arrow. That is, if we randomly pick two targets with arrows, we see that:

1. Each target is hit by exactly one arrow.
2. Each arrow was shot by a different archor.

Property 2 is an immediate consequence of the definition of the function, as each archor only has one arrow to shoot. A function with the above properties is said to be one-to-one, which we formally define as follows:

Definition. A function f that maps A into B is called a **one-to-one function** if no two elements of A have the same value; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2,$$

or equivalently,

$$x_1 = x_2 \qquad \text{whenever } f(x_1) = f(x_2).$$

Now, suppose the targets came to life (!) and decided to stage a revolt (!!) against the tyrant archors. As with the archors, each target is only allowed to shoot one arrow. Then, in order to kill off all archors, it is evident that each target must have exactly one arrow; for otherwise, there would be leftover arrows—which, in turn, implies that there will be survivors. This "shooting-back" procedure is called an *inverse function*, if we require that each target kills the archor with his own arrow. The formal definition is as follows:

Definition. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A. It is defined by:

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

With our construction, it should be clear that

1. $f^{-1}(f(x)) = x$ for every x in A.
2. $f(f^{-1}(x)) = x$ for every x in B.