In this set of notes, we develop a quick way of solving inequalities involving rational expressions. This method depends crucially on the following simple observation:

Theorem.

$$(-1)^n = \begin{cases} 1 & \textit{if } n \textit{ is even;} \\ -1 & \textit{if } n \textit{ is odd.} \end{cases}$$

That is, if you multiply an even number of "negative signs", then you get a "positive sign." Likewise, multiplying an odd number of "negative signs" gives you a "negative sign." We illustrate the method via a series of examples.

Example 1. Let us solve

$$(x-1)(x-2) < 0.$$

There are two "signs" involved here, and we want the product of the signs to be negative. The sign of $(x-1)$ changes at $x = 1$, and the sign of $(x-2)$ changes at $x = 2$, so it suffices to consider three intervals: $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. On the interval $(-\infty, 1)$, both $(x-1)$ and $(x-2)$ are negative, and so the product is positive. On the interval $(1, 2)$, we see that $(x-1)$ is now positive, because $x > 1$. Nevertheless, $(x-2)$ is still negative, for $x < 2$. It thus follows that the product is negative on $(1, 2)$. Finally, both $(x-1)$ and $(x-2)$ are positive on $(2, \infty)$, and so the product is positive.

Since we want the product to be negative, the inequality only holds on the interval $(1, 2)$. Therefore, the correct answer can be written in any of the following forms:

- $1 < x < 2$.
- $\{x \mid 1 < x < 2\}$.
- $(1, 2)$
- x is in $(1, 2)$.

Now, let us solve the same inequality via a different method. We know that the product $(x-1)(x-2)$ is positive on the interval $(2, \infty)$: $x - 2 > 0$ because $x > 2$, and $x - 1 > 0$ because $x > 2$, which implies that $x > 1$. Now, when we move to the next adjacent interval $(1, 2)$, we see that the sign of $(x-2)$ changes, but the sign of $(x-1)$ does not. This means that the expression now has one negative sign, and so the sign of the product is negative. If we, once again, move to the next adjacent interval $(-\infty, 1)$, then the sign of $(x-1)$ changes as well. Now, we have two negative signs, and so the sign of the product is positive. We observe the "flipping" of signs:

- Positive on $(2, \infty)$.
- Negative on $(1, 2)$.
- Positive on $(-\infty, 1)$.

It is this flipping phenomenon that we shall generalize to a wide class of inequalities.

Example 2. Let us now consider the inequality

$$(x-1)(x-2)(3-x) < 0.$$

The "critical points" to consider are $x = 1$, $x = 2$, and $x = 3$, because those are the places where a factor has its sign flipped. On $(3, \infty)$, we have $x - 1 > 0$, $x - 2 > 0$, and $3 - x < 0$, and so:

- Negative on $(3, \infty)$.

If we move to the next adjacent interval $(2, 3)$, then we only change the sign of $(3-x)$: indeed, both $x-1$ and $x-2$ are still positive, but now $3-x$ is positive as well. Since only one sign changed, we see that:

- Positive on $(2, 3)$.

By the same token, moving to the next adjacent interval $(1, 2)$ flips the sign of $(x-2)$ and nothing else. Therefore:

- Negative on $(1, 2)$.

Moving to $(-\infty, 1)$ flips the sign of $(x-1)$, and so:

- Positive on $(-\infty, 1)$.

Since we want the sign of the product to be negative, we only want $(3, \infty)$ and $(1, 2)$. Here are some possible ways of writing down the answer:

- $(1, 2) \cup (3, \infty)$.
- $1 < x < 2$ or $3 < x$.

Example 3. Let us consider the inequality

$$\frac{(x-2)(7-3x)}{(2x-5)(8-3x)(x+1)} > 0.$$

The critical points to consider are $x = 2$, $x = 7/3$, $x = 5/2$, $x = 8/3$, and $x = -1$. In order for our flipping methods to work, we must order the critical points:

$$-1 < 2 < \frac{7}{3} < \frac{5}{2} < \frac{8}{3}.$$

Therefore, the intervals to consider are $(-\infty, -1)$, $(-1, 2)$, $(2, 7/3)$, $(7/3, 5/2)$, $(5/2, 8/3)$, and $(8/3, \infty)$. On $(8/3, \infty)$, $(x-2)$ is positive, $(7-3x)$ is negative, $(2x-5)$ is positive, $(8-3x)$ is negative, and $(x+1)$ is positive. There are two negative signs, so the product is positive. Via flipping, we can now easily deduce the rest:

- Positive on $(8/3, \infty)$.
- Negative on $(5/2, 8/3)$.
- Positive on $(7/3, 5/2)$.
- Negative on $(2, 7/3)$.
- Postiive on $(-1, 2)$.
- Negative on $(-\infty, -1)$.

Since we wanted the whole expression to be positive, we only want $(8/3, \infty)$, $(7/3, 5/2)$, and $(-1, 2)$. Therefore, the answer is:

- $(-1, 2) \cup (7/3, 5/2) \cup (8/3, \infty)$.
- $-1 < x < 2$ or $\frac{7}{3} < x < \frac{5}{2}$ or $\frac{8}{3} < x$.

Example 4. How about

$$\frac{(x-2)(7-3x)}{(2x-5)(8-3x)(x+1)} \geq 0?$$

We simply have to throw in the values of x that will make the expression zero. $x = 2$ and $x = 7/3$ will do. Since $x = 5/2$, $x = 8/3$, and $x = -1$ make the denominator zero, they should *not* be included. Therefore, our new answer should be:

- $(-1, 2] \cup [7/3, 5/2) \cup (8/3, \infty)$.
- $-1 < x \leq 2$ or $\frac{7}{3} \leq x < \frac{5}{2}$ or $\frac{8}{3} < x$.

Example 5. We now discuss an example with *multiplicities*. Consider

$$(x-1)(x-2)^2(x-3) < 0.$$

At $x = 3$, the sign flips once as usual, and so we have:

- Positive on $(3, \infty)$.
- Negative on $(2, 3)$.

Since there are *two* $(x-2)$, the sign flips *twice* at $x = 2$. Indeed, both $(x-2)$ becomes negative on $(1, 2)$. We thus see that

- Negative on $(1, 2)$.

The sign flip once at $x = 1$, and so

- Positive on $(1, 2)$.

Since we want the product to be negative, the answer is

- $(1, 2) \cup (2, 3)$
- $1 < x < 2$ or $2 < x < 3$.

The "flipping" method has a few requirements. First, the inequality should be in the form

`(rational expression) [inequality] 0.`

If we have, for example,

$$\frac{x-1}{x-5} < 3,$$

then we have to get rid of 3 by first substracting 3:

$$\frac{x-1}{x-5} - 3 < 0;$$

and by taking the common denominator:

$$\frac{-2x+14}{x-5} < 0.$$

Second, the method requires the rational expression to be completely factored. This requires practice, practice, and more practice. Here are a few exercises for you to test your understanding:

Exercise 1. Solve
$$(x-1)(x^2-6x+5) > 0.$$

Exercise 2. Solve
$$\frac{x^2-5x+6}{x-5} \leq 0.$$

Exercise 3. Solve
$$\frac{x-2}{4x^2+4x+1} \geq 0.$$