Solutions for Challenge Problem Set 5, Math 291 Fall 2011

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This challenge problem set concerns the effect of friction on ballistic motion: Suppose you are standing at the origin in $\mathbb{R}^3$ and throw a ball with an initial velocity $\mathbf{v}_0$ that we assume to be of the form

$$\mathbf{v}_0 = v_0(\cos\theta, 0, \sin\theta) \tag{0.1}$$

for some angle θ with $0 \le \theta \le \pi/2$. That is, the ball is thrown up and out, along the direction of the positive x-axis.

If you can throw the ball at a maximum speed of v_0 meters per second, what angle should you choose so that the ball travels the farthest before hitting the ground? We have seen that if friction is neglected, the answer is $\theta = \pi/4$ no matter what v_0 is.

Now let us take friction into account. There are various ways to model the effects of friction, depending on the problem at hand, but one way that is suitable for the present problem is to suppose that the frictive force has the direction opposite that of the velocity, with a magnitude proportional to the magnitude of the velocity. That is, friction decelerates, and the greater the velocity, the stronger the frictive force. Thus, for some constant $\alpha > 0$, we suppose the frictive force to be

$$\mathbf{F}_{\text{friction}} = -\alpha\mathbf{v}$$

while the gravitational force is

$$\mathbf{F}_{\text{gravity}} = -mg(0,0,1)\ .$$

Newton's second law then gives

$$m\mathbf{v}'(t) = \mathbf{F}_{\text{friction}} + \mathbf{F}_{\text{gravity}} = -\alpha\mathbf{v}(t) - mg(0,0,1)\ .$$

Let us work in MKS (meter, kilograms and seconds) units. Then we take $g = 9.8m/s^2$. Let us take $m = 1$, and $\alpha = 0.01m/s$. Finally, in (0.1) let us take $v_0 = 30m/s$, which is a bit over 100km/hr.

The velocity vector then satisfies

$$\mathbf{v}'(t) + \frac{1}{100}\mathbf{v}(t) = -9.8(0,0,1)\ . \tag{0.2}$$

1: Multiply both sides of (0.2) by $e^{t/100}$, to obtain

$$e^{t/100}\left(\mathbf{v}'(t) + \frac{1}{20}\mathbf{v}(t)\right) = -e^{t/100}9.8(0,0,1)\ . \tag{0.3}$$

Note that the left hand side is the derivative of $e^{t/100}\mathbf{v}(t)$. Apply the Fundamental Theorem of Calculus; integrate both sides to find an explicit formula for $\mathbf{v}(t)$ in terms of the initial data $\mathbf{v}_0$ as given by (0.1), with $v_0 = 3m/s$.

SOLUTION: The equation now is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/100}\mathbf{v}(t)\right) = -e^{t/100}9.8(0,0,1) \ .$$

Integrating both sides, we find

$$e^{t/100}\mathbf{v}(t) = 30(\cos\theta, 0, \sin\theta) - 9800(e^{t/100} - 1)(0,0,1) \ .$$

Multiplying by $e^{-t/100}$, we get

$$\mathbf{v}(t) = 30e^{-t/100}(\cos\theta, 0, \sin\theta) - 9800(1 - e^{-t/100})(0,0,1) \ .$$

2: Integrate $\mathbf{v}(t) = \mathbf{x}'(t)$ to find and explicit formula for $\mathbf{x}(t)$, recalling that we have chosen $\mathbf{x}(0) = (0,0,0)$ and $\mathbf{v}(0) = 3(\cos\theta, 0, \sin\theta)$ The position at time t also depends on θ, the angle at which the ball is thrown. So you will get a function of two variables, t, and θ: $\mathbf{x}(t,\theta) = (x(t,\theta), 0, z(t,\theta))$.

SOLUTION: From $\mathbf{x}(y) = \int_0^t \mathbf{v}(u)\mathrm{d}u$, since $\mathbf{x}(0) = \mathbf{0}$, we get

$$\mathbf{x}(t) = 3000(1 - e^{-t/100})(\cos\theta, 0, \sin\theta) - 9.8 \times 10^4(t/100 + e^{-t/100} - 1)(0,0,1) \ .$$

The ball hits the ground when $z(t,\theta) = 0$. We want to choose θ in (0.1) to maximize the distance the ball travels before it hits the ground.

• *That is, we seek to maximize $x(t,\theta)$ subject to the constraint $z(t,\theta) = 0$. This is a Lagrange multipliers problem, and we know how to solve it. Let us do so.*

3: Show that the condition $z(t,\theta) = 0$, which specifies the time the ball hits the ground, is equivalent to the equation

$$(\sin\theta + 9800)(1 - e^{-t/100}) = t/20 \ .$$

Show also that

$$x(t,\theta) = 3000\cos\theta(1 - e^{-t/100}) \ .$$

SOLUTION: From the solution to the second part, we have

$$x(t,\theta) = 3000(1 - e^{-t/100})\cos\theta$$

and

$$\begin{aligned} z(t,\theta) &= 3000(1 - e^{-t/100})\sin\theta - 9.8 \times 10^4(t/100 + e^{-t/100} - 1) \\ &= 1000(1 - e^{-t/100})\left[3\sin\theta - 98\left(\frac{t/100}{1 - e^{-t/100}} - 1\right)\right] \end{aligned} \tag{0.4}$$

Thus the equation $z(t,\theta) = 0$ is equivalent to

$$(98 + 3\sin\theta)(1 - e^{-t/100}) = 98t/100 \ .$$

4: Using Lagrange's Theorem, write down the system of equations to be solved to find the maximizers of $x(t,\theta)$ subject to $z(t,\theta) = 0$, using the equivalent formulation of $z(t,\theta) = 0$ from the previous problem.

SOLUTION: We define $g(t,\theta)$ by

$$g(t,\theta) = (98 + 3\sin\theta)(1 - e^{-t/100}) - \frac{98t}{100} \ .$$

Then, taking the gradient in the variables t, θ,

$$\nabla x(t,\theta) = 30\left(\cos\theta e^{-t/100} \ , \ -100\sin\theta(1 - e^{-t/100})\right) \ ,$$

and

$$\nabla g(t,\theta) = \left(\frac{1}{100}((98 + 3\sin\theta)e^{-t/100} - 98) \ , \ 3\cos\theta(1 - e^{-t/100})\right) \ .$$

We then compute (taking out the factor of 3000 in front of $x(t,\theta)$,

$$\frac{1}{3000}\det\left(\begin{bmatrix} \nabla x \\ \nabla g \end{bmatrix}\right) = -\frac{49}{50}(1 - e^{-t/100})^2 + \frac{3}{100}(e^{-t/100} - e^{-t/50}) \ .$$

Thus,

$$\left(\begin{bmatrix} \nabla x \\ \nabla g \end{bmatrix}\right) = 0 \qquad \Longleftrightarrow \qquad -98(1 - e^{-t/100})^2\sin\theta + 3(e^{-t/100} - e^{-t/50}) = 0 \ .$$

Combining this with $g(t,\theta) = 0$, we get the system of equations:

$$\begin{aligned} -98(1 - e^{-t/100})^2\sin\theta + 3(e^{-t/100} - e^{-t/50}) &= 0 \\ (98 + 3\sin\theta)(1 - e^{-t/100}) - 98t/100 &= 0 \ . \end{aligned} \tag{0.5}$$

5: We know the optimal t and θ for the zero friction case. (With zero friction $\theta = \pi/4$, and in the text it is explained how to find the time t when the ball hits the ground given the initial velcoity.) Use these as a starting point, and use Newtons method to compute the optimal angle to three decimal places of accuracy.

SOLUTION: In the case of zero friction and $\theta - \pi/4$, we would have

$$x(t) = \frac{30}{\sqrt{2}}t \qquad \text{and} \qquad \frac{30}{\sqrt{2}}t - \frac{9.8}{2}t^2 \ .$$

We solve $y(t) = 0$ to find $t = 0$ and

$$t = \sqrt{2}\frac{150}{49} \ .$$

Hence we take

$$t_0 = \sqrt{2}\frac{150}{49} \qquad \text{and} \qquad \theta_0 = \frac{\pi}{4} \ .$$

Based on (0.5) define the function $\mathbf{f}(t,\theta)$ by

$$\mathbf{f}(t,\theta) = (\ -98(1-e^{-t/100})^2\sin\theta + 3(e^{-t/100} - e^{-t/50})\ ,\ (98+3\sin\theta)(1-e^{-t/100}) - 98t/100\ )\ .$$

We compute

$$\mathbf{f}(t_0,\theta_0) = (0.002673232, 0.000648474)\ .$$

With

$$\mathbf{x}_0 := (t_0,\theta_0) \approx (4.329225190, 0.7853981635)$$

we then compute

$$\mathbf{x}_1 = \mathbf{x}_0 - [D(\mathbf{x}_0)]^{-1}\mathbf{f}(t_0,\theta_0) = (4.268998367, 0.7784026264)\ .$$

We then evaluate

$$\mathbf{f}(\mathbf{x}_1) = (-0.000044078, -0.000011001)\ .$$

This is pretty good; we are very close to the optimum. Notice that in the presence of friction, the optimal angle is a bit less than $\pi/4$ – about 44.59 degree – and of course the time the ball stays aloft is less too. But not by much – we have uses a small coefficient of friction in this example, and so the effects of friction are a small adjustment.