Challenge Problem Set 2, Math 291 Fall2011

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Let us consider the equation

$$\mathbf{v}'(t) = \mathbf{b}(t) \times \mathbf{v}(t) \quad \text{with} \quad \mathbf{v}(0) = \mathbf{v}_0 \ . \tag{0.1}$$

where $\mathbf{b}(t)$ is a piecewise continuously differentiable function.

Is there a curve that satisfies this equation? The answer is yes, and we shall see how to construct it.

(1) If $\mathbf{b}(t)$ is constant, we know what to do; we have found the explicit solution of (0.1) in this case in the text. Now let

$$\mathbf{b}(t) := \begin{cases} (2,1,2) & 0 < t \leq 1/2 \\ (1,2,2) & 1/2 < t < 1 \end{cases}$$

and let

$$\mathbf{v}_0 := (1,1,1) \ .$$

Find a continuous curve $\mathbf{v}(t)$ for $0 \leq t \leq 1$ that is differentiable except at $t = 1/2$, and which satisfies (0.1) on the interval $(0,1)$ except at $t = 1/2$.

Hint: Do this by using the formulas given in the text to find the solution on $0 \leq t \leq 1/2$. Then use the resulting $\mathbf{v}(1/2)$ as the initial data on the next interval, $1/2 < t \leq 1$, and use the formulas form the text again.

(2) Now suppose $\mathbf{b}(t)$ is piecewise constant with only finitely many discontinuities. Explain why there is a unique continuous curve $\mathbf{v}(t)$ that satisfies (0.1) except at the discontinuities of $\mathbf{b}$. Existence will be pretty clear from what you did in part *(1)*, but still explain it in words. What about uniqueness? Adapt the uniqueness argument given in the text ot cover this more general case.

(3) Now let $\mathbf{b}_1(t)$ and $\mathbf{b}_2(t)$ be two piecewise continuously differentiable curves in $\mathbb{R}^3$ with only finitely many discontinuities. Assume that there exist continuous curves $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ such that for $j = 1,2$, $\mathbf{v}_j(t)$ is differentiable except at the discontinuities of $\mathbf{b}_j(t)$ and such that for $j = 1,2$, and alt where $\mathbf{v}_j$ is differentiable,

$$\mathbf{v}_j'(t) = \mathbf{b}_j(t) \times \mathbf{v}_j(t) \quad \text{with} \quad \mathbf{v}_j(0) = \mathbf{v}_0 \tag{0.2}$$

(We know such curves exist in the piecewise constant case). Notice that the initial condition is the same in both cases.

Show that if $\|\mathbf{b}_1(t) - \mathbf{b}_2(t)\| \le \epsilon$ for all $0 \le t \le 1$, then

$$\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\| \le \epsilon\|\mathbf{v}_0\|$$

for all $0 \le t \le 1$.
Hint: To do this, define $\mathbf{w}(t) := \mathbf{v}_1(t) - \mathbf{v}_2(t)$, and show that

$$\mathbf{w}'(t) = \mathbf{b}_1(t) \times \mathbf{w}(t) + (\mathbf{b}_1(t) - \mathbf{b}_2(t)) \times \mathbf{v}_2(t) \ ,$$

and then use this to derive an equation for $\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|^2$. If you can conclude from this that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|\right| \le \epsilon\|\mathbf{v}_0\| \ ,$$

then you can use the Fundamental Theorem of Calculus, and the fact that $\mathbf{w}(0)\| = 0$, to write

$$\begin{aligned} \|\mathbf{w}(t)\| &= \left|\|\mathbf{w}(0)\| + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{w}(s)\|\mathrm{d}s\right| \\ &\le \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{w}(s)\|\right| \mathrm{d}s \\ &\le \int_0^t \epsilon\|\mathbf{v}_0\|\mathrm{d}s \ . \end{aligned}$$

(4) Extra credit Here is what this is good for. Suppose you want to find and plot the solution to (0.1) for $\mathbf{b}(t)$ that is continuously differentiable. Since continuous functions have maxima on bounded closed intervals, on any such interval, say $[0, 1]$ there will be a C so that

$$\|\mathbf{b}'(t)\| \le C$$

for all $0 \le t \le 1$. But then for any $t > t_0$ in $(0, 1)$,

$$\|\mathbf{v}(t) - \mathbf{v}(t_0)\| = \left\|\int_{t_0}^t \mathbf{b}'(s)\mathrm{d}s\right\| \le \int_{t_0}^t \|\mathbf{b}'(s)\|\mathrm{d}s \le C(t - t_0) \ .$$

Now fix some positive integer k, define a piecewise constant $\mathbf{b}_1(t)$ by

$$\mathbf{b}_1(t) := \mathbf{b}(j2^{-k} \qquad \text{for} \qquad j2^{-k} \le t < (j+1)2^{-k} \ .$$

That is, we chop up $[0, 1]$ into 2^k equal intervals, and in the interior of each interval we set $\mathbf{b}(t)$ equal to the left endpoint of the interval containing t. Define $\mathbf{b}_2(t) := \mathbf{b}(t)$, the given curve. Show that

$$\|\mathbf{b}_1(t) - \mathbf{b}_2(t)\| \le C2^{-k} \ .$$

Use this and previous results to show that the solution $\mathbf{v}_1(t)$ of the approximate equation, which can be constructed explicitly (using a computer if k is large) using the "piecing together" method used in part *(1)*, is very close to the "true" solution for the unmodified $\mathbf{b}(t)$, assuming one exists.
Remark: In fact, one can use these ideas to prove that the solution does exist – the sequence of approximate solutions is a Cauchy sequence in a complete metric space of curves. We will take up these ideas again later in the course. For now, we will satisfy ourselves with showing how to find very accurate approximations to the solutions, assuming they do exist.