Challenge Problem Set 3, Math 291 Fall2011

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This challenge problem set concerns planetary motion, and completes the analysis of Newton's derivation of Kepler's Laws from his Universal Theory of Graviation.

As in the text, a star of mass M is a the center of the coordinate system, and $\mathbf{x}(t)$ denote the position of a planet of mass m at time t. We assume m is much less than M, so that the center of mass of the combined system is effectively the center of the star. Then, the equation of motion of the planet is

$$\mathbf{x}''(t) = -\frac{GM}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t) \ , \tag{0.1}$$

where G is the gravitation constant. Because of the universal nature of Newton's theory, this constant is not only relevany in the heavens. but governs the gravitational attraction between objects on the earth. Therefore, it can be, and has been, measured in laboratories. The value is approximately $G = 6.67 \times 10^{-11}\ Nm^3/kg^2$ in MKS units.

The key to finding the trajectories $\mathbf{x}(t)$ from the differential equation is in the constants of the motion for this equation. Recall that the momentum $\mathbf{p}(t)$, the *angular momentum* $\mathbf{L}(t)$ and the *Runge-Lenz vector* $\mathbf{A}(t)$ of the planet are given by

$$\mathbf{p}(t) = m\mathbf{x}'(t) = m\mathbf{v}(t), \tag{0.2}$$

$$\mathbf{L}(t) = \mathbf{x}(t) \times \mathbf{p}(t) \ . \tag{0.3}$$

and

$$\mathbf{A}(t) = \mathbf{p}(t) \times \mathbf{L}(t) - GMm^2\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} \ . \tag{0.4}$$

Then, if $\mathbf{x}(t)$ solves (0.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}(t) = \mathbf{0} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(t) = \mathbf{0} \ .$$

There is another constant of the motion, namely the *Energy* E given by

$$E(t) = \frac{\|\mathbf{p}(t)\|^2}{2m} - mMG\frac{1}{\|\mathbf{x}(t)\|} \ .$$

(1) Prove that $\frac{\mathrm{d}}{\mathrm{d}t}E = 0$ for solutions of (0.1). Then show that for any solution $\mathbf{x}(t)$ such that the energy E is *negative*, and $\|\mathbf{L}\| \neq 0$, the distance from the center $\|\mathbf{x}(t)\|$ is bounded above and below uniformly for all times t in that there exists constant $0 < r_1 < r_2 < \infty$ so that

$$r_1 \leq \|\mathbf{x}(t)\| \leq r_2 \qquad \text{for all } t \ .$$

since, as explained in the text, the orbits are either ellipses or parabolas, the negative energy orbits are ellipses, and the positive energy orbits are parabolas. (The upper bound is easy to prove by considering the energy E. The lower bound holds whether of not the energy is positive or negative – there is a lower bound for parabolic orbits too – so you get that by considering $\mathbf{A} \cdot \mathbf{x}(t)$.

In the rest of this problem set, we consider a fixed solution of (0.1) with given values of $\mathbf{L}$, $\mathbf{A}$ and $E < 0$.

(2) Let t be such that $\|\mathbf{x}(t)\|' = 0$. Such times will exists since $\|\mathbf{x}(t)\|$ will take on a minimum and a maximum value on the ellipse. Show that at any such time t,

$$\|L\| = \|\mathbf{x}(t)\|\|\mathbf{p}(t)\| \ .$$

Then show that at such a time t, $R := \|\mathbf{x}(t)\|$ satisfies the quadratic equation

$$ER^2 + mMGR - \frac{\|L\|^2}{2m} = 0 \ .$$

(3) Let $R_{\min}$ and $R_{\max}$ be the minimum and maximum values of $\|\mathbf{x}(t)\|$ on the orbit. Show that

$$\|\mathbf{A}\| = m|E|(R_{\max} - R_{\min}) \ .$$

Show that the direction of $\mathbf{A}$ is the unit vector pointing from the origin, which is at a focus of the ellipse, to the point of closest approach of the orbit to the center - the *perihelion* in the case of the Earth and the Sun.

(4) The *eccentricity* of the elliptic orbit is the quantity e defined by

$$e = \frac{R_{\max} - R_{\min}}{R_{\max} + R_{\min}}$$

Express the magnitude of $\mathbf{A}$ in terms of E and e (as well as m, M and G, which are independent of the orbit).