Challenge Problem Set 4, Math 291 Fall 2011

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This challenge problem set concerns *linear independence* and the *rank* of matrices.

Let A be an $m \times n$ matrix written as $A = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of n vectors in $\mathbb{R}^n$. For instance, consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -3 & 0 & -1 \end{bmatrix} ,$$

for which the column vectors, written out horizontaly, are

$$\mathbf{v}_1 = (1, 2, -3) \quad , \quad \mathbf{v}_2 = (-1, 1, 0) \quad \text{and} \quad \mathbf{v}_3 = (2, -1, -1) \ .$$

The *range of* A, $\text{Ran}(A)$ is the set of all vectors in $\mathbb{R}^3$ of the form $A\mathbf{x}$. By the rules of matrix multiplication, this is the set of all vectors of the form

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 \ .$$

That is, $\text{Ran}(A)$ is the set of all linear combination of the columns of A. We now ask:

- *Does* $\text{Ran}(A)$ *"fill up" all of* $\mathbb{R}^3$*? That is, for every* $\mathbf{v} \in \mathbb{R}^3$*, is there some* $\mathbf{x} \in \mathbb{R}^3$ *so that* $A\mathbf{v} = \mathbf{b}$*?*

In this case, the answer is "no". There are several ways to see this. Since this matrix is square, we know it is invertible if and only if its determinant is non-zero. But computing the determinant, we find it is zero. Since the linear transformation given by an $n \times n$ matrix is either invertible or else is neither one-to-one nor onto, the linear transformation given by A is not onto $\mathbb{R}^3$, and so there exist vectors $\mathbf{b}$ for which there is no solutions to $A\mathbf{x} = \mathbf{b}$.

Now consider the 3×4 matrix

$$B = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 \\ -3 & 0 & -1 & -2 \end{bmatrix} ,$$

Then writing $B = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$, where $\mathbf{v}_4 = (1, 1, -2)$, and the $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are again given as above, $\text{Ran}(B)$ is the set of all linear combinations

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 \ .$$

Does Ran(A) "fill up" all of $\mathbb{R}^3$? We have more vectors to work with, however, the linear transformation given by B does not transform $\mathbb{R}^4$ onto $\mathbb{R}^3$. Since B is not square, we cannot see this by computing a determinant. How do we see this, and see for which $\mathbf{b} \in \mathbb{R}^3$ there is a solution to $A\mathbf{x} = \mathbf{b}$?

A very good approach to this problem is to use the Gram-Schmidt orthogonalization procedure to try and build an orthonormal basis for $\mathbb{R}^3$ out of the columns of B. Here is how this goes:

We form

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{14}}(1, 2, -3) \ .$$

This vector is in Ran(B) since it is a multiple of a column of B.

We then try to get a unit vector $\mathbf{u}_2$ in Ran(A) that is orthogonal to $\mathbf{u}_1$ by normalizing the component of $\mathbf{v}_2$ that is orthogonal to $\mathbf{u}_1$, which we can do as long as this component is not zero:

$$(\mathbf{v}_2)_\perp = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1) \cdot \mathbf{u}_1 = (-1, 1, 0) - \frac{1}{14}(1, 2, -3) = \frac{3}{14}(-5, 4, 1) \ .$$

This is not zero, so we can normalize it to obtain

$$\mathbf{u}_2 := \frac{1}{\sqrt{42}}(-5, 4, 1) \ .$$

This vector is in Ran(B) since it is a linear combination of vectors in Ran(B), and therefore a linear combinations of the columns of B. In fact, writing $\mathbf{u}_2$ out in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$ we have

$$\mathbf{u}_2 = \frac{1}{\|(\mathbf{v}_2)_\perp\|}\left(\mathbf{v}_2 - \frac{1}{14}\mathbf{v}_1\right) = -\frac{1}{3\sqrt{42}}\mathbf{v}_1 + \frac{14}{3\sqrt{42}}\mathbf{v}_2 \ .$$

Hence so far, we have found two orthonormal directions in Ran(B).

We now try to get a third orthonormal vector in Ran(B) by taking $\mathbf{v}_3$, and subtracting out the components along $\mathbf{u}_1$ and $\mathbf{u}_2$, getting

$$\mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \ .$$

Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are in Ran(B), and so is $\mathbf{v}_3$, this is a linear combination of vectors in Ran(B), and hence it belongs to Ran(B).

As you can also see, taking dot products with $\mathbf{u}_1$ and $\mathbf{u}_2$, this vector is certainly orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. If it is not zero, we can normalize it to get a third orthonormal vector in Ran(B). However, doing the computations we find

$$\mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 = \mathbf{0} \ .$$

That is,

$$\mathbf{v}_3 = (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \ .$$

The same thing happens with $\mathbf{v}_4$:

$$\mathbf{v}_4 = (\mathbf{v}_4 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}_4 \cdot \mathbf{u}_2)\mathbf{u}_2 \ .$$

Then, since $\mathbf{u}_1$ and $\mathbf{u}_2$ can be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, this means that $\mathbf{v}_3$ and $\mathbf{v}_4$ can be written as linear cominations of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, *the third and fourth columns* *of B do nothing to help "fill out" the range of the linear transformation given by B:* Any linear combination

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4$$

can be reduced to a linear combination of the form

$$y_1\mathbf{v}_1 + y_2\mathbf{v}_2 \ ,$$

simply by expressing $\mathbf{v}_3$ and $\mathbf{v}_4$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ can be written as linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$, by construction, it follows that every vector $\mathbf{b}$ in $\mathrm{Ran}(B)$ can be written as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. In this case, since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthonormal, we must have

$$\mathbf{b} = (\mathbf{b}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{b}\cdot\mathbf{u}_2)\mathbf{u}_2 \ . \tag{0.1}$$

Hence this is a necessary and sufficient condition for $B\mathbf{x} = \mathbf{b}$ to have a solution. We can express this more conveniently as follows: Define $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Then (0.1) is true if and only if

$$\mathbf{b}\cdot\mathbf{u}_3 = 0 \ . \tag{0.2}$$

Computing, we find

$$\mathbf{u}_3 = \frac{1}{\sqrt{3}}(1,1,1) \ .$$

Thus, (0.2) is equivalent to $\mathbf{b}\cdot(1,1,1) = 0$. Hence, for $\mathbf{b} = (a,b,c)$, $B\mathbf{x} = \mathbf{b}$ has a solution if and only if $a+b+c=0$.

You can easily check that each of the columns of B satisfy this equation. Indeed, since $\mathbf{v}_j = B\mathbf{e}_j$, the equation $B\mathbf{x} = \mathbf{v}_j$ has a solution, namely $\mathbf{e}_j$, and so $\mathbf{v}_j$ must be orthogonal to $(1,1,1)$ by what we have noted above.

Conversely, had we noted that for each $j = 1,2,3,4$,

$$\mathbf{v}_j \cdot (1,1,1) = 0 \ ,$$

by the distributivte properties of the dot product we would conclude

$$(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4)\cdot(1,1,1) = 0 \ .$$

Hence, $\mathrm{Ran}(A)$ is contained in the plane given by $x+y+z=0$. Since $\mathrm{Ran}(A)$ contains a set of two orthonormal vectors, $\{\mathbf{u}_1, \mathbf{u}_2\}$, it contains a plane, and must be exactly equal to this plane.

• *Hence the Gram-Schmidt procedure allows us to determine exactly what is, and what is not, in the range of a linear transformation.*

We now make some key definitions that let us conveniently summarize what we have seen in this example:

The *rank* of a linear transformation is the maximal size of an orthonormal set contained in the range of the transformation, which is therefore the dimension of the range. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is linearly independent if and only if

$$\sum_{j=1}^{k} x_j\mathbf{v}_j = \mathbf{0} \qquad \Longleftrightarrow \qquad x_1 = x_2 = \ldots, x_k = 0 \ .$$

In particular, when $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, it is impossible to express $\mathbf{v}_1$ as a linear combination of $\{\mathbf{v}_2, \ldots, \mathbf{v}_k\}$ since if this were the case, we would have

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k \ ,$$

but then

$$\mathbf{v}_1 - a_2\mathbf{v}_2 - \cdots - a_k\mathbf{v}_k = \mathbf{0} \ ,$$

and not all the coefficients are zero – note the coefficient of $\mathbf{v}_1$ is 1. By symmetry in the indices, the same is true of the other vectors; none can be expressed as a linear combination of the others. A bit of thought shows that this not only implied by linear independence, but it is equivalent to it.

So we can check for linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ by applying the Gram-Schmidt procedure: If at the ℓth stage we get a zero vector when computing the component of $\mathbf{v}_\ell$ that is orthogonal to $\{\mathbf{u}_1, \ldots, \mathbf{u}_{\ell-1}\}$, i.e., if

$$\mathbf{v}_\ell - \sum_{j=1}^{\ell-1}(\mathbf{v}_\ell \cdot \mathbf{u}_j)\mathbf{u}_j = \mathbf{0} \ ,$$

then $\mathbf{v}_\ell$ can be written as a linear combination of the vectors in $\{\mathbf{v}_1, \ldots, \mathbf{v}_{\ell-1}\}$, and so $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is *not* linearly independent. If we discard such vectors, and keep going with the rest, we get an orthonormal basis of $\text{Ran}([\mathbf{v}_1, \ldots, \mathbf{v}_k])$, and the number of vectors in this basis is the rank of $[\mathbf{v}_1, \ldots, \mathbf{v}_k]$.

Exercise 1: Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -2 & 0 \end{bmatrix} .$$

(a) Use the Gram-Schmidt procedure to find an orthonormal basis of $\text{Ran}(A)$.

(b) What is the rank of A?

(c) Are the columns of A linearly independent?

(d) Find an equation for $\text{Ran}(A)$; i.e., an equation that is satisfied by $\mathbf{b} \in \mathbb{R}^3$ if and only if $\mathbf{b} \in \text{Ran}(A)$.

(e) For $\mathbf{v} = (2, 1, 1)$, does $A\mathbf{x} = \mathbf{b}$ have a solution?

Exercise 2: Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & -2 & -2 \\ -2 & 1 & 0 & 1 \end{bmatrix} .$$

(a) Use the Gram-Schmidt procedure to find an orthonormal basis of $\text{Ran}(A)$.

(b) What is the rank of A?

(c) Are the columns of A linearly independent?

(d) Find an equation for $\text{Ran}(A)$; i.e., an equation that is satisfied by $\mathbf{b} \in \mathbb{R}^3$ if and only if $\mathbf{b} \in \text{Ran}(A)$.

(e) For $\mathbf{v} = (1, -4, 1)$, does $A\mathbf{x} = \mathbf{b}$ have a solution?

Exercise 3: Let $A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$ be a 2×3 matrix. Show that the rank of A is 2 if and only if $\mathbf{a}_1 \times \mathbf{a}_2 \neq \mathbf{0}$.