Challenge Problem Set 6, Math 291 Fall 2011

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November 21, 2011

This challenge problem set concerns the effect moments of inertia and motion involving the rotation of a rigid body. Let $\mathcal{V}$ be the subset of $\mathbb{R}^3$ occupied by an object that is a rigid body, and suppose that this object has a mass density function $\varrho(\mathbf{x})$, so that the total mass of the object is

$$M = \int_{\mathcal{V}} \varrho(\mathbf{x}) \mathrm{d}V \ .$$

Now suppose that the object is set in motion so that it rotates about the line through the origin along the the direction $\mathbf{u}$ where $\mathbf{u}$ is some unit vector in $\mathbb{R}^3$. Suppose that the angular velocity of the rotation (in radians per second) is ω.

By what we have learned about the rotation equation in Chapter 2, each point in the object moves along a circular trajectory that is satisfies the equation

$$\mathbf{x}'(t) = \boldsymbol{\omega} \times \mathbf{x}(t)$$

where $\boldsymbol{\omega} = \omega \mathbf{u}$. (See Theorem 26 in particular.)

Therefore, the total kinetic energy of the rotating object at time t is

$$T = \frac{1}{2} \int_{\mathcal{V}} \varrho(\mathbf{x}) \|\boldsymbol{\omega} \times \mathbf{x}\|^2 \mathrm{d}V \tag{0.1}$$

where one integrates over the region occupied by the object at time t. (For simplicity, we have not indicated in our notation that $\mathcal{V}$ depends on t, which is will unless of course the object is rotationally symmetric bout the axis of rotation.)

1. Use the cross product identities we proved earlier in the course to show that

$$\|\boldsymbol{\omega} \times \mathbf{x}\|^2 = -\boldsymbol{\omega} \cdot (\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega})) \ ,$$

and also show that for fixed $\mathbf{x}$, the transformation sending $\boldsymbol{\omega}$ to $-\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega})$ is the linear tranformation given by the matrix

$$\begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} .$$

2. By the result of Exercise **1.**, the formula (0.1) becomes

$$T = \frac{1}{2}\int_{\mathcal{V}} \varrho(\mathbf{x}) \left(\boldsymbol{\omega} \cdot \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \boldsymbol{\omega} \right) \mathrm{d}V \tag{0.2}$$

Use the fact that $\boldsymbol{\omega}$ a constant vector to show that this is the same as

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} \tag{0.3}$$

where

$$\mathbf{I} := \begin{bmatrix} \int_{\mathcal{V}}(y^2+z^2)\varrho \mathrm{d}V & -\int_{\mathcal{V}} xy\varrho \mathrm{d}V & -\int_{\mathcal{V}} xz\varrho \mathrm{d}V \\ -\int_{\mathcal{V}} xy\varrho \mathrm{d}V & \int_{\mathcal{V}}(x^2+z^2)\varrho \mathrm{d}V & -\int_{\mathcal{V}} yz\varrho \mathrm{d}V \\ -\int_{\mathcal{V}} xz\varrho \mathrm{d}V & -\int_{\mathcal{V}} yz\varrho \mathrm{d}V & \int_{\mathcal{V}}(x^2+y^2)\varrho \mathrm{d}V \end{bmatrix} \tag{0.4}$$

is the *moment of inertia matrix* of the body. (Do not confuse it with the identity matrix!)

2. Let $\mathcal{V}$ be the region lying above the cone given by $z = \sqrt{x^2+y^2}$, and inside the sphere $x^2+y^2+z^2 = 4$. Suppose this is occupied by an object that has a mass density

$$\varrho(x,y,z) = x^2+y^2+z^2 \ .$$

Compute the moment of inertia matrix $\mathbf{I}$ for this object. Suppose that the object is set rotating at unit angular velocity about some the line through the origin with direction vector $\mathbf{u}$, where $\mathbf{u}$ is some unit vector. Which choices of $\mathbf{u}$ make the kinetic energy largest? Which choices of $\mathbf{u}$ make the kinetic energy smallest?

4. Consider the region in the x, z plane that lies. above $z = -1$, below $z = 1$ and between the z axis and the curve $x = 1 + z^2$. Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ obtained by rotation this planar region bout the z-xis to produce a solid of revolution.

Let us consider two objects occupying the region region $\mathcal{V}$. The first has a uniform mass density with total mass M. The second one has its mass concentrated uniformly on the curved surface of the object (so none is in the caps at either end), and again the total mass is M.

Calculate $\mathbf{I}_{3,3}$ for both of these objects, which is what is relevant to rotation about their axis of symmetry, and which is what is therefore relevant to problems concerning rolling such objects down an incline.

5. Consider again the two objects from Exercise **4**. Suppose they are simultaneously let go on a plane that is inclined at an angle of $\pi/4$ with the horizontal, so that they start rolling down under the influence of gravity. Let us take the distances along the axes to be measured in meters, and take the gravitational constant to be 9.8 m/s^2. As explained in class, one may use the conservation of energy, our knowledge of rotational and translational kinetic energy to compute the height that the center of mass will have come down in time t, form which you can figure out anything about the motion.

Compute how far along the incline that each object will have rolled at time $t = 10$ seconds. (Note that the value of M will drop out, so you do not need this value, and it does not even matter that the two masses are the same.)