Challenge Problem Set 7, Math 291 Fall 2011

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0.1 Introduction

Let φ be a twice continuously differentiable function defined on $\mathbb{R}^3$ with values in $\mathbb{R}$. The Laplacian of φ is the function $\Delta\varphi$ defined by

$$\Delta\varphi(\mathbf{x}) = \frac{\partial^2}{\partial x^2}\varphi(\mathbf{x}) + \frac{\partial^2}{\partial y^2}\varphi(\mathbf{x}) + \frac{\partial^2}{\partial y^2}\varphi(\mathbf{x}) \ .$$

Note that

$$\Delta\varphi(\mathbf{x}) = \mathrm{div}(\nabla\phi(\mathbf{x})) \ .$$

Let ρ be a continuous function defined on $\mathbb{R}^3$ with values in $\mathbb{R}$. *Poisson's Equation* is the equation

$$\Delta\varphi(\mathbf{x}) = \rho(\mathbf{x}) \ .$$

Think of ρ as a given function. We then seek to to find all twice continuously differentiable functions φ such that $\Delta\varphi(\mathbf{x}) = \rho(\mathbf{x})$.

In many ways, this problem is analogous to the basic problem of linear algebra: Given a vector $\mathbf{b} \in \mathbb{R}^m$ and an $m \times n$ matrix A, the linear algebra problem is to find all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$. The analogy is a good one because taking the Laplacian of a function is a linear operation: Since differentiation is linear, if φ and ψ are two twice continuously differentiable functions, and a and b two numbers, then

$$\Delta(a\varphi + b\psi)(\mathbf{x}) = a\Delta\varphi(\mathbf{x}) + b\Delta\psi(\mathbf{x})$$

for all $\mathbf{x}$.

0.2 Solving Poisson's equaation

Our goal is to use the vector calculus we have learned to solve Poisson's Equation. The key to this is the following observation due to George Green: Let φ and ψ be two twice continuously differentiable real valued functions on an open set $U \subset \mathbb{R}^3$. Define the vector field $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{x}) := \varphi(\mathbf{x})\nabla\psi(\mathbf{x}) - \psi(\mathbf{x})\nabla\varphi(\mathbf{x}) \ . \tag{0.1}$$

Then, as shown in class,

$$\mathrm{div}\mathbf{F}(\mathbf{x}) = \varphi(\mathbf{x})\Delta\psi(\mathbf{x}) - \psi(\mathbf{x})\Delta\varphi(\mathbf{x}) \tag{0.2}$$

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everywhere in U.

Now fix $\mathbf{y} \in \mathbb{R}^3$, and define the function

$$\psi(\mathbf{x}) = \frac{1}{\|\mathbf{x} - \mathbf{y}\|}$$

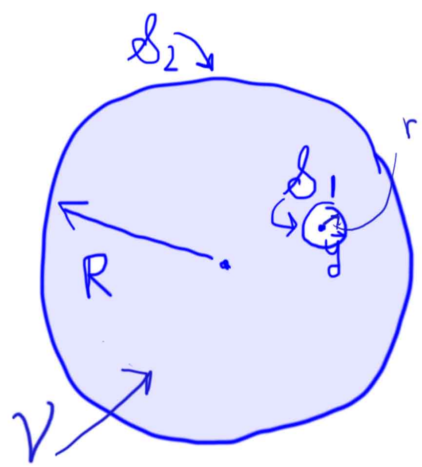
on the set $U = \mathbb{R}^3 \backslash \{\mathbf{y}\}$; i.e., $\mathbb{R}^3$ with $\mathbf{y}$ removed. Then as shown in class,

$$\nabla \psi(\mathbf{x}) = -\frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^3} \qquad \text{and} \qquad \Delta \psi(\mathbf{x}) = 0$$

everywhere in U. Hence with this choice of ψ, (0.2) becomes

$$\text{div}\mathbf{F}(\mathbf{x}) = -\frac{1}{\|\mathbf{x} - \mathbf{y}\|} \Delta \varphi(\mathbf{x}) \ . \tag{0.3}$$

Now let r and R be positive numbers such that the ball of radius r centered at $\mathbf{y}$ is contained in the ball of radius R centered at $\mathbf{0}$. Let $\mathcal{V}$ be the region that is inside the ball of radius R centered at $\mathbf{0}$, and outside the ball of radius r centered at $\mathbf{y}$. The boundary of $\mathcal{V}$ consists of two spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ where $\mathcal{S}_1$ denotes the sphere of radius r centered at $\mathbf{y}$, and $\mathcal{S}_2$ denotes the sphere of radius R centered at $\mathbf{0}$.



Then by the divergence Theorem,

$$\int_{\mathcal{V}} \text{div}\mathbf{F}(\mathbf{x}) \mathrm{d}V = \int_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S + \int_{\mathcal{S}_2} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S \ .$$

By (0.3),

$$\lim_{R \to \infty} \lim_{r \to 0} \int_{\mathcal{V}} \text{div}\mathbf{F}(\mathbf{x}) \mathrm{d}V = -\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \Delta \varphi(\mathbf{x}) \mathrm{d}V \ .$$

1: Show that

$$\lim_{r \to 0} \int_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S = 4\pi \varphi(\mathbf{y}) \ .$$

For this, use the fact that

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= -\varphi(\mathbf{x}) \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^3} - \nabla \varphi(\mathbf{x}) \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \\ &= \frac{1}{r^2} \varphi(\mathbf{x}) \mathbf{N} - \frac{1}{r} \nabla \varphi(\mathbf{x}) \end{aligned}$$

everywhere on $\mathcal{S}_1$, together with the fact that the total surface area of $\mathcal{S}_1$ is $4\pi r^2$.

2: Now suppose that φ is such that

$$\int_{\mathbb{R}^3} |\Delta\varphi(\mathbf{x})|\mathrm{d}V < \infty \ .$$

In the context of Poisson's equation $\Delta\varphi = \rho$ with ρ being a charge density, this means that the total absolute charge is finite.

Use the fact that everywhere on $\mathcal{S}_2$, $\psi(\mathbf{x}) = 1/R$, and so

$$\int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \frac{1}{R}\int_{\mathcal{S}_2} \nabla\varphi(\mathbf{x})\cdot\mathbf{N}\mathrm{d}S \ ,$$

and then one more application of the Divergence Theorem, to conclude

$$\left|\int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S\right| \le \frac{1}{R}\int_{\mathbb{R}^3} |\Delta\varphi(\mathbf{x})|\mathrm{d}V \ ,$$

and then show that

$$\lim_{R\to\infty}\int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 0 \ .$$

3: Combine the above to prove the following theorem:

0.1 THEOREM. *Let φ be a twice continuously differentiable function on $\mathbb{R}^3$ such that*

$$\int_{\mathbb{R}^3} |\Delta\varphi(\mathbf{x})|\mathrm{d}V < \infty \ .$$

Then for all $\mathbf{y} \in \mathbb{R}^3$,

$$\varphi(\mathbf{y}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\Delta\varphi(\mathbf{x})\mathrm{d}V \ . \tag{0.4}$$

This theorem suggests that when ρ is such that $\int_{\mathbb{R}^3} |\rho(\mathbf{x})|\mathrm{d}V < \infty$, then $\Delta\varphi(\mathbf{x}) = \rho(\mathbf{x})$ has a solution given by

$$\varphi(\mathbf{y}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\rho(\mathbf{x})\mathrm{d}V \ .$$

This is the case, as the next exercises show, at least when ρ itself is twice continuously differentiable.

To see this, define the function ϕ by

$$\phi(\mathbf{y}) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\rho(\mathbf{x})\mathrm{d}V \ . \tag{0.5}$$

Note that, for any fixed $\mathbf{y}$, if we make a change of variables by introducing the new variables $\mathbf{z} := \mathbf{y}-\mathbf{x}$, which is simply a translation followed by multiplication by -1, the absolute Jacobian determinant for this change of variables is simply 1 everywhere. Since $\mathbf{x} = \mathbf{y}-\mathbf{z}$, we have

$$\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\rho(\mathbf{x})\mathrm{d}V = \int_{\mathbb{R}^3} \frac{1}{\|\mathbf{z}\|}\rho(\mathbf{y}-\mathbf{z})\mathrm{d}V \ , \tag{0.6}$$

where on the left $\mathbf{x}$ is the variable of integration, while on the right $\mathbf{z}$ is the variables of integration.

Hence, using (0.6), we see that (0.5) is equivalent to

$$\phi(\mathbf{y}) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{z}\|}\rho(\mathbf{y}-\mathbf{z})\mathrm{d}V \ . \tag{0.7}$$

Now let us take the Laplacian of both sides in the $\mathbf{y}$ variables. Since this is a linear operation, we may take these derivatives under the integral sign (the integral is in $\mathbf{z}$, not $\mathbf{y}$) and we see

$$\Delta\phi(\mathbf{y}) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{z}\|}\Delta\rho(\mathbf{y}-\mathbf{z})\mathrm{d}V \ .$$

Using (0.6) with ρ replaced by $\Delta\rho$, we see that this is equivalent to

$$\Delta\phi(\mathbf{y}) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\Delta\rho(\mathbf{x})\mathrm{d}V \ . \tag{0.8}$$

Comparing this with (0.4) in Theorem 0.1, in which we replace φ by ρ, we see that

$$\Delta\phi(\mathbf{y}) = \rho(\mathbf{y})$$

for all $\mathbf{y}$. Thus, (0.6) does indeed give us a solution of Poisson's equation.

0.3 Laplace's equation and uniqueness for Poisson's Equation

Consider the functions

$$\phi_1(\mathbf{x}) = x \qquad \phi_2(\mathbf{x}) = 2x^2 - y^2 - z^2 \quad \text{and} \quad \phi_3(\mathbf{x}) = \cos(x)e^y \ .$$

As you can easily check, $\Delta\phi_j(\mathbf{x}) = 0$ for all $\mathbf{x}$ for each $j = 1, 2, 3$. That is, each of these functions is a solutions of Laplace's equation $\Delta\phi(\mathbf{x}) = 0$.

Now suppose that for some given density ρ, we have a solution φ of Poisson's equation: $\Delta\varphi = \rho$. Then, since taking the Laplacian is a linear operation,

$$\Delta(\varphi + \phi_j) = \Delta\varphi + \Delta\phi_j(\mathbf{x}) = \rho + 0 = \rho$$

for each j. Hence solutions of Poisson's equation are never unique: You can take any solution, and add to it any solution of Laplace's equation, and you get another solution.

However, the solution given by the formula

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\rho(\mathbf{x})\mathrm{d}V \ . \tag{0.9}$$

is special: It is the only solution that has the property that

$$\lim_{\|\mathbf{x}\|\to\infty} \varphi(\mathbf{x}) = 0 \ . \tag{0.10}$$

Let us first show that our special solution does have this property:

4: Assume not only that ρ is twice continuously differentiable, but also that for some $R > 0$, $\rho(\mathbf{x}) = 0$ for all $\|\mathbf{x}\| > R$. That is, there is no "charge" or "mass" associated to our density ρ

outside a ball of some finite radius. This assumption can be relaxed, but it is reasonable enough, and easy to work with, so we make it.

Show that under this assumption, $\int_{\mathbb{R}^3} |\rho(\mathbf{x})| \mathrm{d}V < \infty$ and also, for $\|\mathbf{y}\| > 2R$,

$$\left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \rho(\mathbf{x}) \mathrm{d}V \right| \le \frac{1}{2\pi \|\mathbf{y}\|} \int_{\mathbb{R}^3} |\rho(\mathbf{x})| \mathrm{d}V \ ,$$

and hence that under these assumptions on ρ, the solution of Poisson's equation given by (0.9) satisfies (0.10).

5: Let some density ρ be given, and suppose that for this density we have two solutions φ_1 and φ_2 of Poisson's equation both of which satisfy (0.10). Show that their difference $\phi := \varphi_1 - \varphi_2$ is a solutions of Laplace's equation which satisfies (0.10).

In the rest of this problem set, our goal is to prove that the only solution ϕ of Laplace's equation which satisfies (0.10) is the zero function $\phi(\mathbf{x}) = 0$ for all $\mathbf{x}$. Once we have proved this, it will follow that the two solutions φ_1 and φ_2 of Poisson's equation considered in Exercise 5 are actually the same, and thus, by the result of Exercise 4, the solution of Poisson's equation given by (0.9) is the unique solution that satisfies (0.10). This turns out to be the only physically meaningful solution in many applications: Roughly, as you get very far away from a finite mass or charge density, its effects should go to zero.

Fix any $\mathbf{y} \in \mathbb{R}^3$, and let $\psi(\mathbf{x}) = 1/\|\mathbf{y} - \mathbf{x}\|$ as above. Let ϕ be any solution of Laplace's equation, and let

$$\mathbf{F}(\mathbf{x}) := \phi(\mathbf{x}) \nabla \psi(\mathbf{x}) - \psi(\mathbf{x}) \nabla \phi(\mathbf{x}) \ ,$$

so that, by Green's identity $\mathrm{div}\mathbf{F} = \mathbf{0}$ everywhere in $U := \mathbb{R}^2 \backslash \{\mathbf{y}\}$. Now for any $0 < r < R < \infty$, let $\mathcal{V}$ be the region in $\mathbb{R}^3$ consisting of points $\mathbf{x}$ with

$$r < \|\mathbf{x}\| < R \ .$$

The boundary of this region is the union of two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ where $\mathcal{S}_1$ is the sphere of radius r centered at $\mathbf{y}$, and $\mathcal{S}_2$ is the sphere of radius R centered at $\mathbf{y}$. The diagram is just like the previous diagram, except that now both spheres are centered at the origin.

Then by the Divergence Theorem,

$$0 = \int_{\mathcal{V}} \mathrm{div}\mathbf{F}(\mathbf{x}) \mathrm{d}V = \int_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S + \int_{\mathcal{S}_2} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S \ . \tag{0.11}$$

6: Referring to (0.11), show first that

$$\nabla \psi = \frac{1}{r^2} \mathbf{N} \qquad \text{and} \qquad \nabla \psi = \frac{1}{r}$$

everywhere on $\mathcal{S}_1$, while

$$\nabla \psi = -\frac{1}{R^2} \mathbf{N} \qquad \text{and} \qquad \nabla \psi = \frac{1}{R}$$

everywhere on $\mathcal{S}_2$. Use this and (0.11) to show that

$$\frac{1}{r^2} \int_{\mathcal{S}_1} \varphi \mathrm{d}S - \frac{1}{R^2} \int_{\mathcal{S}_2} \varphi \mathrm{d}S = \frac{1}{r} \int_{\mathcal{S}_1} \nabla \varphi \cdot \mathbf{N} \mathrm{d}S + \frac{1}{R} \int_{\mathcal{S}_2} \nabla \varphi \cdot \mathbf{N} \mathrm{d}S \ .$$

Then use the Divergence Theorem once more to show that

$$\int_{\mathcal{S}_1} \nabla\varphi \cdot \mathbf{N}\mathrm{d}S = \int_{\mathcal{S}_2} \nabla\varphi \cdot \mathbf{N}\mathrm{d}S = 0 \ .$$

Conclude that

$$\frac{1}{4\pi r^2}\int_{\mathcal{S}_1} \varphi\mathrm{d}S = \frac{1}{4\pi R^2}\int_{\mathcal{S}_2} \varphi\mathrm{d}S \ .$$

Since the total surface area of $\mathcal{S}_1$ is $4\pi r^2$, and since the total surface area of $\mathcal{S}_2$ is $4\pi R^2$, the quantity on the left is the average value of φ over the sphere of radius r centered at $\mathbf{y}$, while the quantity on the right is the average value of φ over the sphere of radius R centered at $\mathbf{y}$.

7: Prove the following Theorems:

0.2 THEOREM (Mean value property)**.** *Let φ be a twice continuously differentiable function on $\mathbb{R}^3$ that satisfies Laplace's equation $\Delta\varphi = 0$. Then for all $\mathbf{y} \in \mathbb{R}^3$ and all $R > 0$,*

$$\varphi(\mathbf{y}) = \frac{1}{4\pi R^2}\int_{\mathcal{S}_R(\mathbf{y})} \varphi\mathrm{d}S$$

where $\mathcal{S}_R(\mathbf{y})$ is the sphere of radius R centered at $\mathbf{y}$. That is, for any solution φ of Laplace's equation on $\mathbb{R}^3$, the value of φ at any point $\mathbf{y}$ is equal to the average value of ϕ on the sphere of any positive radius R about that point.

0.3 THEOREM (Liouville's Theorem)**.** *The only solution φ of Laplace's equation $\Delta\varphi = 0$ on all of $\mathbb{R}^3$ that has the property that $\lim_{\|\mathbf{x}\|\to\infty} \varphi(\mathbf{x}) = 0$ is the zero function $\varphi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$.*

0.4 THEOREM (Existence and uniqueness for Poisson's Equation)**.** *Let ρ be a twice continuously differentiable function on $\mathbb{R}^3$ such that $\rho(\mathbf{x}) = 0$ for all $\mathbf{x}$ outside some bounded subset of $\mathbb{R}^3$. Then there exists a unique solution of $\Delta\varphi = \rho$ such that $\lim_{\|\mathbf{y}\|\to\infty} \varphi(\mathbf{y}) = 0$, and this solution is given by the integral formula*

$$\varphi(\mathbf{y}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x} - \mathbf{y}\|}\rho(\mathbf{x})\mathrm{d}V \ .$$

Note that the condition "all of $\mathbb{R}^3$" in Theorem 0.3 is essential: As we have seen, the function $\psi(\mathbf{x}) := 1/\|\mathbf{x} - by\|$ satisfies Laplace's equations everywhere on $U := \mathbb{R}^3\backslash\{\mathbf{y}\}$, but not on all of $\mathbb{R}^3$. The difference may be only one point, but it matters.