MATH 291 WORKSHOP 3

MARK H. KIM

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1. Suggested Solutions to the Challenge Problems

We first recall that $0 < m \ll M$, $G = 6.67 \times 10^{-11}\, Nm^3/kg^2$, and $\boldsymbol{x}(t) : [0, \infty) \to \mathbb{R}^3$ satisfies the differential equation

$$\boldsymbol{x}''(t) = -\frac{GM}{\|\boldsymbol{x}(t)\|^3}\boldsymbol{x}(t).$$

Furthermore, if

$$\begin{aligned} \boldsymbol{p}(t) &= m\boldsymbol{x}'(t), \\ \boldsymbol{L}(t) &= \boldsymbol{x}(t) \times \boldsymbol{p}(t), \\ \boldsymbol{A}(t) &= \boldsymbol{p}(t) \times \boldsymbol{L}(t) - GMm^2\frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|}, \end{aligned}$$

then

$$\frac{d}{dt}\boldsymbol{L}(t) = \frac{d}{dt}\boldsymbol{A}(t) = \boldsymbol{0}.$$

Since $\boldsymbol{L}(t)$ and $\boldsymbol{A}(t)$ are constant, we shall drop the t and write $\boldsymbol{L}$ and $\boldsymbol{A}$ instead.

1.1. Problem 1.

We set

$$E(t) = \frac{\|\boldsymbol{p}(t)\|^2}{2m} - mMG\frac{1}{\|\boldsymbol{x}(t)\|}.$$

We claim that $\frac{d}{dt}E(t) = 0$. To this end, we observe first that

$$\frac{d}{dt}E(t) = \frac{\|\boldsymbol{p}(t)\|}{m}\left(\frac{d}{dt}\|\boldsymbol{p}(t)\|\right) + \frac{mMG}{\|\boldsymbol{x}(t)\|^2}\left(\frac{d}{dt}\|\boldsymbol{x}(t)\|\right). \tag{1}$$

Every differentiable vector-valued function $\boldsymbol{v}(t) : \mathbb{R} \to \mathbb{R}^3$ satisfies the identity

$$\frac{d}{dt}\|\boldsymbol{v}(t)\| = \frac{d}{dt}\sqrt{\boldsymbol{v}(t) \cdot \boldsymbol{v}(t)} = \frac{\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t)}{\sqrt{\boldsymbol{v}(t) \cdot \boldsymbol{v}(t)}} = \frac{\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t)}{\sqrt{\|\boldsymbol{v}(t)\|}},$$

and so equation (1) reduces to

$$\frac{d}{dt}E(t) = \frac{\boldsymbol{p}(t)\cdot\boldsymbol{p}'(t)}{m} + \frac{mMG(\boldsymbol{x}(t)\cdot\boldsymbol{x}'(t))}{\|\boldsymbol{x}(t)\|^3}. \tag{2}$$

Since

$$\begin{aligned} \boldsymbol{p}(t) &= m\boldsymbol{x}'(t), \\ \boldsymbol{p}'(t) &= m\boldsymbol{x}''(t) = -\frac{mMG}{\|\boldsymbol{x}(t)\|^3}\boldsymbol{x}(t), \end{aligned}$$

equation (2) reduces to

$$\frac{d}{dt}E(t) = -\frac{mMG}{\|\boldsymbol{x}(t)\|^3}(\boldsymbol{x}(t)\cdot\boldsymbol{x}'(t)) + \frac{mMG}{\|\boldsymbol{x}(t)\|^3}(\boldsymbol{x}(t)\cdot\boldsymbol{x}'(t)) = 0. \tag{3}$$

It follows that $E(t)$ is constant everywhere. Henceforth, we shall drop the variable and denote $E(t)$ by E.

We now suppose that $E < 0$ and $\|\boldsymbol{L}\| \neq 0$. for all $t \in [0, \infty)$. We claim that there exists constants $0 < r_1 < r_2 < \infty$ such that

$$r_1 \leq \|\boldsymbol{x}(t)\| \leq r_2$$

for all $t \in [0,\infty)$. We first note that

$$\frac{mMG}{\|\boldsymbol{x}(t)\|} = \frac{\|\boldsymbol{p}(t)\|^2}{2m} - E \geq -E > 0,$$

and so

$$\|\boldsymbol{x}(t)\| \leq -\frac{mMG}{-E}.$$

To obtain the lower bound, we first observe the identity

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{x}(t) &= (\boldsymbol{x}(t)\times\boldsymbol{p}(t))\cdot\boldsymbol{L} - GMm^2\|\boldsymbol{x}(t)\| \\ &= \|\boldsymbol{L}\|^2 - GMm^2\|\boldsymbol{x}(t)\|. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\|\boldsymbol{L}\|^2 = \boldsymbol{A}\cdot\boldsymbol{x}(t) + GMm^2\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{A}\|\|\boldsymbol{x}(t)\| + GMm^2\|\boldsymbol{x}(t)\|,$$

whence a simple rearrangement yields

$$\|\boldsymbol{x}(t)\| \geq \frac{\|L(t)\|^2}{\|A(t)\| + GMm^2}.$$

1.2. **Problem 2.** We henceforth assume that $E < 0$. Find $t_0 \in [0, \infty)$ such that $\|\boldsymbol{x}(t_0)\|' = 0$[1]. Since

$$\frac{d}{dt}\|\boldsymbol{x}(t)\| = \frac{\boldsymbol{x}(t)\cdot\boldsymbol{x}'(t)}{\|\boldsymbol{x}(t)\|},$$

this implies that $\boldsymbol{x}(t_0)\cdot\boldsymbol{x}'(t_0) = 0$. This, in particular, shows that $\boldsymbol{x}(t_0)$ and $\boldsymbol{p}(t_0)$ are orthogonal, whence

$$\|\boldsymbol{L}(t_0)\| = \|\boldsymbol{x}(t_0)\|\|\boldsymbol{p}(t_0)\|\sin(90^\circ) = \|\boldsymbol{x}(t_0)\|\|\boldsymbol{p}(t_0)\|.$$

Setting $R(t_0) = \|\boldsymbol{x}(t_0)\|$, we have

$$E = \frac{\|\boldsymbol{p}(t_0)\|^2}{2m} - mMG\frac{1}{\|\boldsymbol{x}(t_0)\|} = \frac{\|\boldsymbol{L}(t_0)\|^2}{2mR(t_0)^2} - \frac{mMG}{R(t_0)},$$

[1]The existence of such t_0 follows from the compactness of the ellipse.

or

$$ER(t_0)^2 + mMGR(t_0) - \frac{\|\boldsymbol{L}(t_0)\|^2}{2m} = 0. \tag{4}$$

1.3. **Problem 3.** Let $R_{\max} = \max_t \|\boldsymbol{x}(t)\|$ and $R_{\min} = \min_t \|\boldsymbol{x}(t)\|$. Since equation (4) is satisfied at all critical points of $\|\boldsymbol{x}(t)\|$, $R_{\max}$ and $R_{\min}$ are solutions of equation (4). Since equation (4) has at most two solutions, we have

$$ER(t)^2 + mMGR(t) - \frac{\|\boldsymbol{L}\|^2}{2m} = E(R(t) - R_{\max})(R(t) - R_{\min}), \tag{5}$$

where t ranges over all critical points of $\|\boldsymbol{x}(t)\|$. By the quadratic formula, we have

$$R_{\max} - R_{\min} = \frac{\sqrt{m^2M^2G^2 - \frac{4\|\boldsymbol{L}\|^2 E}{2m}}}{|E|},$$

or

$$m|E|(R_{\max} - R_{\min}) = \sqrt{m^4M^2G^2 + 2m\|\boldsymbol{L}\|^2 E}.$$

Since t ranges over the critical points of $\|\boldsymbol{x}(t)\|$, we have $\|\boldsymbol{p}(t)\| = m\|\boldsymbol{x}'(t)\| = 0$ for all such t. Therefore, $\|\boldsymbol{L}\| = 0$ for all such t, and we have

$$m|E|(R_{\max} - R_{\min}) = \sqrt{m^4M^2G^2}.$$

For these t, we also have

$$\|\boldsymbol{A}\| = \left\|\boldsymbol{p}(t) \times \boldsymbol{L} - GMm^2\frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|}\right\| = \left\|-GMm^2\frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|}\right\| = \sqrt{m^4M^2G^2},$$

whence

$$\|\boldsymbol{A}\| = m|E|(R_{\max} - R_{\min})$$

for the same t. Since the above identity no longer depends on t, the qualification "for all t ranging over the critical points of $\|\boldsymbol{x}(t)\|$" is unnecessary.

We claim that the direction of $\boldsymbol{A}$ is the unit vector pointing from the origin to the point of closest approach of the orbit to the center. Upon fixing $t_1 \in [0, \infty)$ such that $\|\boldsymbol{x}(t_1)\| = R_{\min}$, this claim is equivalent to the identity

$$\frac{\boldsymbol{A}}{\|\boldsymbol{A}\|} = \frac{\boldsymbol{x}(t_1)}{\|\boldsymbol{x}(t_1)\|}.$$

Indeed, we observe that

$$\boldsymbol{A} = \boldsymbol{p}(t_1) \times \boldsymbol{L}(t_1) - GMm^2\frac{\boldsymbol{x}(t_1)}{R_{\min}} = \boldsymbol{p}(t_1) \times (\boldsymbol{x}(t_1) \times \boldsymbol{p}(t_1)) - GMm^2\frac{\boldsymbol{x}(t_1)}{R_{\min}}.$$

We have shown that $\boldsymbol{x}(t)$ and $\boldsymbol{p}(t)$ are orthogonal at the critical points of $\|\boldsymbol{x}(t)\|$, so

$$\boldsymbol{p}(t_1)\times(\boldsymbol{x}(t_1)\times\boldsymbol{p}(t_1)) = \boldsymbol{x}(t_1)(\boldsymbol{p}(t_1)\cdot\boldsymbol{p}(t_1)) - \boldsymbol{p}(t_1)(\boldsymbol{p}(t_1)\cdot\boldsymbol{x}(t_1)) = \boldsymbol{x}(t_1)(\boldsymbol{p}(t_1)\cdot\boldsymbol{p}(t_1)).$$

Therefore,

$$\boldsymbol{A} = \left(\|\boldsymbol{p}(t_1)\|^2 - \frac{GMm^2}{R_{\min}}\right)\boldsymbol{x}(t_1),$$

whence the claim follows.

1.4. **Problem 4.** Equation (5) implies that

$$R_{\max} + R_{\min} = -\frac{GMm}{E},$$

and so

$$\|\boldsymbol{A}\| = m|E|(R_{\max} - R_{\min}) = m|E|e(R_{\max} + R_{\min}) = eGMm^2.$$