# HONORS CALUCLUS I, FALL 2015 - A LOG INEQUALITY

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Let us make the intuition of

`polynomials grow faster than the logarithm`

precise. Here is our first attempt:

**Theorem 1.** *If* $p \geq 1$, *then*

$$x^p > \ln x$$

*for all* $x \geq 1$.

*Proof.* Fix $p \geq 1$ and let $f(x) = x^p$ and $g(x) = \ln x$. We first note that

$$f(1) = 1 > 0 = g(1).$$

Now, $p \geq 1 > 0$, and so $p - 1 > -1$. Therefore,

$$f'(x) = px^{p-1} \tag{1.1}$$

$$\geq x^{p-1} \tag{1.2}$$

$$\geq x^{-1} = g'(x) \tag{1.3}$$

for all $x \geq 1$. $f(1) > g(1)$, and $f$ grows faster than $g$ on $[1, \infty)$, hence we conclude that

$$f(x) > g(x)$$

for all $x \in [1, \infty)$, as was to be shown. □

The same proof does not go through for $0 < p < 1$, as inequality (1.2) fails to hold. The following modification, however, suffices for many applications:

**Theorem 2.** *If* $p > 0$, *then*

$$\frac{1}{p}x^p \geq \ln x$$

*for all* $x \geq 1$.

*Proof.* Fix $p > 0$ and let $f(x) = \frac{1}{p}x^p$ and $g(x) = \ln x$. Since $p > 0$, we note that

$$f(1) = \frac{1}{p} > 0 = g(1).$$

Now, $p > 0$, and so $p - 1 > -1$. Therefore

$$f'(x) = x^{p-1} \tag{2.1}$$

$$\geq x^{-1} = g'(x) \tag{2.2}$$

for all $x \geq 1$. $f(1) > g(1)$, and $f$ grows faster than $g$ on $[1, \infty)$, hence we conclude that

$$f(x) > g(x)$$

for all $x \in [1, \infty)$, as was to be shown. □

Note that the first two steps (1.1 - 1.2) in Theorem 1 are combined into one step (2.1) in Theorem 2. Indeed, the added coefficient $p$ for $f$ in Theorem 2 makes this possible.

We also remark that Theorem 2 is stronger than Theorem 1 in the $p > 1$ case, as

$$\frac{1}{p}x^p < x^p$$

for all $x \geq 1$.

*Example* 3. As a simple application, we show that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^r}$$

converges for each $r > 1$. Fix $r > 1$, so that $r - 1 > 0$. Theorem 2 implies that

$$\frac{\ln n}{n^r} < \frac{1}{pn^{r-p}}$$

for each $p > 0$ and every $n \geq 1$. We pick $p \in (0, r-1)$ so that $r - p > 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{pn^{r-p}} = \frac{1}{p} \sum_{n=1}^{\infty} \frac{1}{n^{r-p}}$$

converges, and so

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^r}$$

converges by the comparison test. □

*Example* 4. We shall show that

$$\sum_{n=2}^{\infty} \frac{1}{n^r \ln n}$$

diverges for all $0 < r < 1$. Fix $r \in (0, 1)$, so that $1 - r > 0$. Theorem 2 implies that

$$\frac{1}{n^r \ln n} > \frac{p}{n^{r+p}}$$

for each $p > 0$ and every $n \geq 1$. Pick $p \in (0, 1 - r)$, so that $p + r < 1$. Therefore,

$$\sum_{n=2}^{\infty} \frac{p}{n^{r+p}} = p \sum_{n=2}^{\infty} \frac{1}{n^{r+p}}$$

diverges, and so

$$\sum_{n=2}^{\infty} \frac{1}{n^r \ln n}$$

diverges by the comparison test. □