HONORS CALCULUS I, FALL 2015 - BESSEL FUNCTIONS

For each nonnegative integer r, we define the *Bessel function of order r* to be

$$J_r(x) = \left(\frac{x}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}.$$

1. Bessel Function of Order 0

We first study the $r = 0$ case:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

This is a power series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $x_0 = 0$, $a_{2n-1} = 0$ for all $n \geq 1$, and

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

1.1. **Computation of the Radius of Convergence.** Set

$$b_n(x) = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

for each $n \geq 0$, so that

$$J_0(x) = \sum_{n=0}^{\infty} b_n(x).$$

Observe that, for a fixed $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2} / 2^{2n+2}((n+1)!)^2}{(-1)^n x^{2n} / 2^{2n}(n!)^2} \right| \\ &= \lim_{n\to\infty} \left| \frac{-x^2}{2^2(n+1)^2} \right| = 0\end{aligned}$$

regardless of the choice of $|x|$. The ratio test implies that J_0 converges absolutely for all $x \in \mathbb{R}$. □

Date: October 2, 2015.

1.2. **A Cautionary Remark.** We remark that it is impractical to use the power-series radius of convergence formulas

$$R = \left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} = \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\right)^{-1}$$

directly, as the odd terms a_{2n+1} are zero. We can nevertheless use the ratio test or the root test directly to the series

$$\sum_{n=0}^{\infty} b_n(x)$$

for each fixed x.

1.3. **Root Test?** We could, of course, apply the root test to

$$\sum_{n=0}^{\infty} b_n(x)$$

as well. Let's see what we get:

$$\lim_{n\to\infty} \sqrt[n]{|b_n(x)|} = \lim_{n\to\infty} \sqrt[n]{\frac{|(-1)^n x^{2n}|}{2^{2n}(n!)^2}} = \lim_{n\to\infty} \frac{|x|^2}{4(n!)^{2/n}}.$$

It would appear that we need to know how to compute

$$\lim_{n\to\infty} (n!)^{2/n}.$$

To this end, we observe that a simple arrangement yields the following:

$$(n!)^2 = (n \cdot 1)((n-1)\cdot 2)((n-2)\cdot 3)\cdots(3\cdot(n-2))(2\cdot(n-1))(1\cdot n).$$

In other words,

$$(n!)^2 = \prod_{k=1}^{n} (n+1-k)k.$$

Now, for each $1 \leq k \leq n$,

$$(n+1-k)k - n = nk - n + k - k^2 = n(k-1) + k(k-1) = (n+k)(k-1) \geq 0,$$

and so

$$(n+1-k) \geq n.$$

It follows that

$$(n!)^2 = \prod_{k=1}^{n} (n+1-k)k \geq \prod_{k=1}^{n} n = n^n,$$

whence it follows that

$$\lim_{n\to\infty} (n!)^{2/n} \geq \lim_{n\to\infty} n = \infty.$$

Returning to the task at hand, we see that

$$\lim_{n\to\infty} \sqrt[n]{|b_n(x)|} = \lim_{n\to\infty} \frac{|x|^2}{4(n!)^{2/n}} = 0$$

regardless of the choice of x. The root test therefore implies that J_0 converges absolutely for all x.

1.4. **Bessel's Differential Equation of Order 0.** We now show that

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0$$

for all $x \in \mathbb{R}$. To this end, we observe that

$$\begin{aligned} J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1} n!(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2(n+1)-1}}{2^{2(n+1)-1}(n+1)!((n+1)-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+1} n!(n+1)!}. \end{aligned}$$

Observe that $J_0'(x)$ does not have a constant term, and so the computation of $J_0''(x)$ does not eliminate the $n = 0$ term:

$$J_0''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1) x^{2n}}{2^{2n+1} n!(n+1)!}.$$

Now, for each $x \in \mathbb{R}$,

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n+1} n!(n+1)!} [2(n+1) - 1 - (2n+1)] = 0,$$

as was to be shown. □

2. Bessel Functions of Order r

In this section, we shall compute the radius of convergence of J_r for a fixed positive integer r. We shall also show that $y = J_r(x)$ is a solution to *Bessel's differential equation*

$$x^2 y'' + xy' + (x^2 - r^2)y = 0.$$

2.1. **Computation of the Radius of Convergence.** The $\left(\frac{x}{2}\right)^r$ term is irrelevant in computing the radius of convergence: it suffices to investigate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}.$$

We set

$$b_n = \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}$$

for each $n \geq 0$ and observe that

$$\begin{aligned} \left|\frac{b_{n+1}}{b_n}\right| &= \left|\frac{(-1)^{n+1} x^{2n+2}/2^{2+2n}(n+1)!(n+r+1)!}{(-1)^n x^{2n}/2^{2n} n!(n+r)!}\right| \\ &= \left|\frac{-x^2}{4(n+1)(n+r+1)}\right| = \frac{|x|^2}{4(n+1)(n+r+1)}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = 0 < 1$$

regardless of our choice of x, the ratio test implies that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}$$

converges absolutely on $(-\infty, \infty)$. It follows that the radius of convergence of J_r is ∞.

2.2. **Bessel's Differential Equation of Order r.** Now,

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r},$$

and $r \geq 1$, and so J_r has no constant term. Therefore,

$$J_r'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)}{2(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-1}.$$

If $r = 1$, then

$$J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{2(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n},$$

and so $J_1'(x)$ has a constant term. Therefore,

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)(2n)}{4(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n-1}$$

If $r > 1$, then

$$J_r'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)}{2(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-1}$$

does not have a constant term, and so

$$J_r''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)(2n+r-1)}{4(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-2}.$$

Note, however, that $r = 1$ implies that

$$\frac{(-1)^n(2n+r)(2n+r-1)}{4(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-2} = \frac{(-1)^n(2n+1)(2n)}{4(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n-1}.$$

When $n = 0$, we obtain the following:

$$\frac{(-1)^n(2\cdot 0+1)(2\cdot 0)}{4(0!)(0+1)!} \left(\frac{x}{2}\right)^{2\cdot 0-1} = 0.$$

It follows that

$$J_r''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)(2n+r-1)}{4(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-2}$$

in the $r = 1$ case as well, despite the presence of the $\frac{1}{x}$ term. In other words, there is no need to consider the $r = 1$ case and the $r > 1$ case separately.

Let us now verify that $J_r(x)$ satisfies Bessel's differential equation

$$x^2 J_r''(x) + xJ_r'(x) + (x^2 - r^2)J_r(x) = 0.$$

Since

$$x^2 J_r''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+r)(2n+r-1)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}$$

$$x J_r'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}$$

$$(x^2 - r^2) J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2 - r^2)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r},$$

we see that

$$\begin{aligned}
&x^2 J_r''(x) + x J_r'(x) + (x^2 - r^2) J_r(x) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} \big((2n+r)(2n+r-1) + (2n+r) + (x^2 - r^2)\big) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} \left(4n^2 + 4nr + x^2\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=0}^{\infty} \frac{(-1)^n x^2}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}.
\end{aligned}$$

Now, when $n = 0$,

$$\frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} = \frac{(-1)^0 4 \cdot 0 \cdot (0+r)}{0!(0+r)!} \left(\frac{x}{2}\right)^{2 \cdot 0 + r},$$

and so

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} = \sum_{n=1}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}.$$

It now follows that

$$\begin{aligned}
&x^2 J_r''(x) + x J_r'(x) + (x^2 - r^2) J_r(x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=0}^{\infty} \frac{(-1)^n x^2}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(n-1)!(n+r-1)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=0}^{\infty} \frac{(-1)^n 4}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r+2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(n-1)!(n+r-1)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(n-1)!(n+r-1)!} \left(\frac{x}{2}\right)^{2n+r} = 0,
\end{aligned}$$

as was to be shown. □