HONORS CALCULUS I, FALL 2015 - QUIZ 2 SOLUTIONS

Problem 1 (Strang 3.2.15)**.** Suppose an m by n matrix has r pivots. The number of special solutions is ________. The nullspace contains only $x = 0$ when $r =$ ________. The column space is all of $\mathbb{R}^m$ when $r =$ ________.

Solution. The number of special solutions, which equals the nullity of the matrix, is $n - r$. (Recall that `rank + nullity` $= n$.) The nullspace contains only $x = 0$ when the nullspace is trivial, which is when the nullity $n - r$ is 0. In other words, this happens when $r = n$. The column space is a subspace of $\mathbb{R}^m$ whose dimension is r, the number of pivots. Therefore, the column space is all of $\mathbb{R}^m$ when $r = m$, the maximum possible dimension. □

Problem 2 (Strang 3.1.19)**.** Describe the column spaces of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution. A has only one pivot, so the rank of A is 1. Therefore, the column space $C(A)$ must be spanned by one vector. Indeed,

$$C(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\},$$

which is a line in $\mathbb{R}^2$.

B has two pivots, so the rank of B is 2. This simply means that

$$C(B) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ 2y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\},$$

which is a plane in $\mathbb{R}^3$.

C has one pivot, so the rank of C is 1. Therefore, the column space $C(C)$ must be spanned by one vector. Now, the zero vector does not contribute to the span, and so

$$C(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} : x, x \in \mathbb{R} \right\},$$

which is a line in $\mathbb{R}^3$. □

Problem 3 (Strang 3.2.26; Strang 3.2.30)**.** Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(a) Show that $N(A) = C(A)$.
(b) Show that $N(A) \neq N(A^T)$.

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(c) Show that $(\text{rref}(A))^T \neq \text{rref}(A^T)$.

Solution. (a) $N(A)$ is the collection of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix},$$

we see that $N(A)$ is the collection of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $y = 0$, i.e.,

$$N(A) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}.$$

$C(A)$ is the span of all column vectors of A:

$$C(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}.$$

It follows that $C(A) = N(A)$.

(b) Proceed as in (a), noting that

$$A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

□