HONORS CALCULUS I, FALL 2015 - QUIZ 3 SOLUTIONS

Problem 1. State the definitions of the following concepts:

(1) linear dependence
(2) linear independence
(3) span of a set of vectors
(4) basis
(5) null space of a matrix
(6) column space of a matrix
(7) row space of a matrix
(8) left nullspace of a matrix
(9) rank of a matrix
(10) nullity of a matrix

Solutions. (1) A set $\{v_1, \ldots, v_n\}$ of vectors is said to be *linearly dependent* if there exist scalars $\lambda_1, \ldots, \lambda_n$ such that $\lambda_i \neq 0$ for some $1 \leq i \leq n$ and

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0.$$

Equivalently, $\{v_1, \ldots, v_n\}$ is linearly dependent if there exists $1 \leq i \leq n$ such that

$$\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, v_n\}.$$

(2) A set $\{v_1, \ldots, v_n\}$ of vectors is said to be *linearly independent* if

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$$

implies that $\lambda_1 = \cdots = \lambda_n = 0$. Equivalently, $\{v_1, \ldots, v_n\}$ is linearly independent if, for each $1 \leq i \leq n$,

$$\operatorname{span}\{v_1, \ldots, v_n\} \supsetneq \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, v_n\}.$$

(3) The *span* of a set $\{v_1, \ldots, v_n\}$ of vectors is the set of all linear combinations of $v_1, \ldots, v_n$:

$$\left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_1, \ldots, \lambda_n \in \mathbb{R} \right\}.$$

(4) A *basis* of a linear subspace M of $\mathbb{R}^m$ is a linearly independent subset of M whose span is M. We remark that every basis of M has the same number of vectors—this number is called the *dimension* of M.

(5) Let A be an m-by-n matrix. The *nullspace* of A is the linear subspace

$$\mathcal{N}(A) = \{v \in \mathbb{R}^n : Av = 0\}$$

of $\mathbb{R}^n$.

(6) Let A be an m-by-n matrix. The *column space* of A is the linear subspace $\mathcal{C}(A)$ of $\mathbb{R}^m$ spanned by the column vectors of A.

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(7) Let A be an m-by-n matrix. The *row space* of A is $\mathcal{C}(A^T)$, the column space of the transpose of A. Equivalently, the row space of A is the linear subspace of $\mathbb{R}^n$ spanned by the row vectors of A.

(8) Let A be an m-by-n matrix. The *left nullspace* of A is $\mathcal{N}(A^T)$, the nullspace of the transpose of A^T. Equivalently, the left nullspace of A is the linear subspace

$$\{v \in \mathbb{R}^m : v^T A = 0^T\}.$$

(9) Let A be an m-by-n matrix. The *rank* of A is the dimension of $\mathcal{C}(A)$, the column space of A. Equivalently, the rank of A is the dimension of $\mathcal{C}(A^T)$, the row space of A. □

Problem 2 (Strang 3.5.38)**.** Which of the following are bases for $\mathbb{R}^3$?

(1) $(1,2,0)$ and $(0,1,-1)$
(2) $(1,1,-1)$, $(2,3,4)$, $(4,1,-1)$, $(0,1,-1)$
(3) $(1,2,2)$, $(-1,2,1)$, $(0,8,0)$
(4) $(1,2,2)$, $(-1,2,1)$, $(0,8,6)$

Solutions. Let M be an m-dimensional linear subspace of $\mathbb{R}^n$, and let $B \subseteq M$. We recall that B spans M only if B contains at least m vectors. We also recall that B cannot contain more than m vectors if B is linearly independent. Therefore, if B is a basis for M, then B must contain m vectors.

Now, $\mathbb{R}^3$ is a 3-dimensional subspace of $\mathbb{R}^3$. Therefore, (1) and (2) cannot be bases for $\mathbb{R}^3$.

We note that three vectors v_1, v_2, v_3 form basis of $\mathbb{R}^3$ if and only if the column space of the matrix $[v_1 \mid v_2 \mid v_3]$ is $\mathbb{R}^3$. Since the column space of a 3-by-3 matrix is $\mathbb{R}^3$ if and only if its reduced row echelon form is the 3-by-3 identity matrix, it suffices to form matrices with vectors in (3) and (4) and check the reduced row echelon forms of those matrices.

For (3), the reduced row echelon form is computed as follows:

$$\begin{bmatrix}1 & -1 & 0\\ 2 & 2 & 8\\ 2 & 1 & 0\end{bmatrix} \xrightarrow{-2r_1+r_2} \begin{bmatrix}1 & -1 & 0\\ 0 & 4 & 8\\ 2 & 1 & 0\end{bmatrix} \xrightarrow{-2r_1+r_3} \begin{bmatrix}1 & -1 & 0\\ 0 & 4 & 8\\ 0 & 3 & 0\end{bmatrix}$$
$$\xrightarrow{\frac{1}{4}r_2} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 2\\ 0 & 3 & 0\end{bmatrix} \xrightarrow{r_2+r_1} \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 3 & 0\end{bmatrix}$$
$$\xrightarrow{-3r_2+r_3} \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & -6\end{bmatrix} \xrightarrow{-\frac{1}{6}r_3} \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$$
$$\xrightarrow{-2r_3+r_1} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} \xrightarrow{-2r_3+r_2} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Since the reduced row echelon matrix is the identity matrix, we see that (3) is a basis for $\mathbb{R}^3$.

As for (4), the reduced row echelon form is computed as follows:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 6 \end{bmatrix} \xrightarrow{-2r_1+r_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 8 \\ 2 & 1 & 6 \end{bmatrix} \xrightarrow{-2r_1+r_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 8 \\ 0 & 3 & 6 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{4}r_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{r_2+r_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$
$$\xrightarrow{-3r_2+r_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the reduced row echelon matrix is not the identity matrix, we see that (4) is not a basis for $\mathbb{R}^3$. □

Problem 3. Define the four fundamental subspaces of a matrix. State the fundamental theorem of linear algebra, part 1 (p. 188 in Strang). Given an m-by-n matrix A, what is the sum of the rank of A and the nullity of A?

Solution. The four fundamental subspaces of a matrix are: the column space, the row space, the nullspace, and the left nullspace. By the fundamental theorem of linear algebra,

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

□

Problem 4 (Strang 3.6.14)**.** Without computing A, find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Solution. This is the LU decomposition of A, with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Since L is the product of elementary matrices, L is invertible. In other words, $Lv = 0$ if and only if $v = 0$. Therefore, $LUx = 0$ if and only if $Ux = 0$, whence the nullspace of A equals the nullspace of U. To compute the nullspace of U, we write

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and determine a, b, c, and d:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a + 2b + 3c + 4d \\ b + 2c + 3d \\ c + 2d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{array}{ccccc} d & = & d & & \\ c & = & -2d & & \\ b & = & -2c - 3d & = d \\ a & = & -2b - 3c - 4d & = 0. \end{array}$$

It follows that the nullspace of A is

$$\operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} : \lambda \in \mathbb{R} \right\},$$

which is a one-dimensional linear subspace of $\mathbb{R}^4$.

By the rank-nullity theorem (Problem 3), the rank of A is 3. Since the column space $\mathcal{C}(A)$ of A is a linear subspace of $\mathbb{R}^3$, we conclude that $\mathcal{C}(A)$ must be $\mathbb{R}^3$. Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of $\mathcal{C}(A)$.

Since $\dim \mathcal{C}(A^T) = \dim \mathcal{C}(A)$, the row space $\mathcal{C}(A^T)$ must be a 3-dimensional linear subspace of $\mathbb{R}^4$. By the rank-nullity theorem, the nullity of A^T is 0. Therefore, the left nullspace of A^T has empty basis.

Finally, to exhibit a basis of $\mathcal{C}(A^T)$, we shall show that the row space of A equals the row space of U. To this end, it suffices to show that the operation of adding a multiple of one row to another row does not affect the row space. In other words,

$$\operatorname{span}\{v_1, v_2, v_3, \ldots, v_n\} = \operatorname{span}\{v_1, av_1 + v_2, v_3, \ldots, v_n\}$$

for all $a \in \mathbb{R}^n$. Indeed,

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = (\lambda_1 - a\lambda_2)v_1 + \lambda_2(av_1 + v_2) + \cdots + \lambda v_n,$$

and so every linear combination of $v_1, v_2, v_3, \ldots, v_n$ is a linear combination of $v_1, av_1 + v_2, v_3, \ldots, v_n$. Conversely,

$$\lambda_1 v_1 + \lambda_2(av_1 + v_2) + \cdots + \lambda_n v_n = (\lambda_1 + a\lambda_2)v_1 + \lambda v_2 + \cdots + \lambda_n v_n,$$

and so every linear combination of $v_1, av_1 + v_2, v_3, \ldots, v_n$ is a linear combination of $v_1, v_2, v_3, \ldots, v_n$. This completes the proof.

Now, $\mathcal{C}(A^T) = \mathcal{C}(U^T)$, and so the row space of A is

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Since U is in the reduced form, the three vectors are already linearly independent. It follows that

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

form a basis for the row space of A. $\square$