HONORS CALCULUS I, FALL 2015 - QUIZ 4 SOLUTIONS

Problem 1. Compute the determinant of the following 5-by-5 matrices.

$$(a)\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}\quad(b)\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\2&2&2&2&0\\1&1&1&1&1\end{bmatrix}\quad(c)\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&1&0\\1&1&1&0&0\\1&1&1&1&1\end{bmatrix}$$

(d) Given that

$$\det\begin{bmatrix}1&0&0&0&0\\1&1&0&4&4\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}=1,$$

compute the determinant of

$$\begin{bmatrix}1&0&0&0&0\\2&2&0&6&6\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}$$

Solutions. (a) By cofactor expansion:

$$\begin{aligned}\det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}=&1\cdot\det\begin{bmatrix}1&0&2&2\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}-0\cdot\det\begin{bmatrix}1&0&2&2\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\\&+0\cdot\det\begin{bmatrix}1&1&2&2\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}-0\cdot\det\begin{bmatrix}1&1&0&2\\1&1&1&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\\&+0\cdot\det\begin{bmatrix}1&1&0&2\\1&1&1&0\\1&1&1&1\\1&1&1&1\end{bmatrix}=\det\begin{bmatrix}1&0&2&2\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}.\end{aligned}$$

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$$\det\begin{bmatrix}1&0&2&2\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix} = 1\cdot\det\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} - 0\cdot\det\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} + 2\det\begin{bmatrix}1&1&0\\1&1&0\\1&1&1\end{bmatrix} - 2\det\begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix} = \det\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix},$$

because the determinant of a matrix with repeated rows is 0.

$$\det\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} = 1\cdot\det\begin{bmatrix}1&0\\1&1\end{bmatrix} - 0\cdot\det\begin{bmatrix}1&0\\1&1\end{bmatrix} + 0\cdot\det\begin{bmatrix}1&1\\1&1\end{bmatrix} = 1.$$

It follows that

$$\det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix} = 1.$$

(b) Since det is multilinear,

$$\det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\2&2&2&2&0\\1&1&1&1&1\end{bmatrix} = \det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\2\cdot1&2\cdot1&2\cdot1&2\cdot1&0\\1&1&1&1&1\end{bmatrix} = 2\det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\2&2&2&2&0\\1&1&1&1&1\end{bmatrix} = 2\cdot1 = 2.$$

(c) Since det is alternating,

$$\det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&1&0\\1&1&1&0&0\\1&1&1&1&1\end{bmatrix} = -\det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix} = -1.$$

(d) Since det is multilinear,

$$\det\begin{bmatrix}1&0&0&0&0\\2&2&0&6&6\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix} = \det\begin{bmatrix}1&0&0&0&0\\1&1&0&2&2\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix} + \det\begin{bmatrix}1&0&0&0&0\\1&1&0&4&4\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}$$
$$= 1 + 1 = 2.$$

□

Problem 2. Recall that the *cross product* of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is the (symbolic) determinant

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}.$$

Recall also that the *dot product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\boldsymbol{u} \bullet \boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Show that the *triple product* $(\boldsymbol{u} \times \boldsymbol{v}) \bullet \boldsymbol{w}$ of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w} = (w_1, w_2, w_3)$ is the determinant

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

What is the significance of this number?

Solution. The triple product represents the volume of the parallelepiped generated by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.. Note that

$$\begin{aligned} \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= (u_2v_3 - u_3v_1)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\ &= (u_2v_3 - u_3v_1, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \cdot (w_1, w_2, w_3) \\ &= (\boldsymbol{u} \times \boldsymbol{v}) \bullet \boldsymbol{w}. \end{aligned}$$

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