Section 1.3, Problem 29

On the surface, this problem is a simple application of the special product formula (page 26):

$$(A+B)(A-B) = A^2 - B^2.$$

That is,

$$(x^2-a^2)(x^2+a^2) = (x^2)^2 - (a^2)^2 = x^4 - a^4.$$

This, in fact, implies that

$$x^4 - a^4 = (x^2-a^2)(x^2+a^2),$$

if we were to factor the expression $x^4 - a^4$. We are not done yet, however; $x^2 - a^2$ is another "difference of two squares," viz.,

$$(x^2-a^2)(x^2+a^2) = (x-a)(x+a)(x^2+a^2).$$

Let us observe the special factoring formulas (page 29)

$$A^2 - B^2 = (A-B)(A+B)$$

and

$$A^3 - B^3 = (A-B)(A^2+AB+B^2).$$

Following the pattern, it seems reasonable to guess that

$$A^4 - B^4 = (A-B)(A^3 + A^2B + AB^2 + B^3).$$

Indeed, we note that

$$(x^4-a^4) = (x^2-a^2)(x^2+a^2) = (x-a)(x+a)(x^2+a^2) = (x-a)(x^3+x^2a+a^2x+a^3).$$

In general, the formula for factoring an expression of the form $A^n - B^n$ is the following:

$$A^n - B^n = (A-B)(A^n + A^{n-1}B + A^{n-2}B^2 + \cdots + A^2B^{n-2} + AB^{n-1} + B^n).$$

See Problems 111 and 112 on page 33 for related problems.

Section 1.3, Problem 94

This is an exercise in factoring by grouping.

$$\begin{aligned} x^3+3x^2-x-3 &= (x^3+3x^2)-(x+3) \\ &= x^2(x+3)-(x+3) \\ &= x^2(x+3)-1(x+3) \\ &= (x^2-1)(x+3) \\ &= (x+1)(x-1)(x+3) \end{aligned}$$

As a challenge, try factoring $x^3+y^3+z^3-3xyz$ (this is *very* difficult).

Section 1.4, Problem 51

This is a problem we went over in class, but the technique used here is worth documenting.

Note that the definition of fraction is

$$\frac{A}{B} = A \div B.$$

This is to say that we can always partition a complicated compound fractional expression into two pieces—the numerator, and the denominator. We may then simplify them separately, and combine them back via division operation. Let us look at the problem:

$$\frac{\frac{x}{y} - \frac{y}{x}}{\frac{1}{x^2} - \frac{1}{y^2}}$$

Group the top, group the bottom, divide:

$$\left(\frac{x}{y} - \frac{y}{x}\right) \div \left(\frac{1}{x^2} - \frac{1}{y^2}\right)$$

Let us simplify the numerator first.

$$\begin{aligned}
\left(\frac{x}{y} - \frac{y}{x}\right) \div \left(\frac{1}{x^2} - \frac{1}{y^2}\right) &= \left(\frac{x^2}{xy} - \frac{y^2}{xy}\right) \div \left(\frac{1}{x^2} - \frac{1}{y^2}\right) \\
&= \left(\frac{x^2 - y^2}{xy}\right) \div \left(\frac{1}{x^2} - \frac{1}{y^2}\right) \\
&= \left(\frac{(x+y)(x-y)}{xy}\right) \div \left(\frac{1}{x^2} - \frac{1}{y^2}\right)
\end{aligned}$$

And now, to the denominator:

$$\begin{aligned}
\left(\frac{(x+y)(x-y)}{xy}\right) \div \left(\frac{1}{x^2} - \frac{1}{y^2}\right) &= \left(\frac{(x+y)(x-y)}{xy}\right) \div \left(\frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2}\right) \\
&= \left(\frac{(x+y)(x-y)}{xy}\right) \div \left(\frac{y^2 - x^2}{x^2y^2}\right) \\
&= \left(\frac{(x+y)(x-y)}{xy}\right) \div \left(\frac{(y-x)(y+x)}{x^2y^2}\right)
\end{aligned}$$

We apply the "flip the fraction and make it a multiplication" trick:

$$\left(\frac{(x+y)(x-y)}{xy}\right) \div \left(\frac{(y-x)(y+x)}{x^2y^2}\right) = \frac{(x+y)(x-y)}{xy} \times \frac{x^2y^2}{(y-x)(y+x)}$$

Let us check what we can cancel out. $x + y = y + x$, hence they clearly cancel out. Furthermore, $x^2y^2 = (xy)(xy)$, so we can cancel out one of the xy. Therefore,

$$\frac{(x+y)(x-y)}{xy} \times \frac{x^2y^2}{(y-x)(y+x)} = \frac{(x-y)(xy)}{y-x}$$

Are we done? Not quite. Observe that $(x-y) = -(y-x)$ (If you need to be convinced, expand the expression on the right). This immediately implies that

$$\frac{(x-y)(xy)}{y-x} = \frac{xy}{-1} = -xy$$

Section 1.4, Problem 71

This problem tests the basic understanding of computations involving fractional and negative exponents. Here are two important rules to keep in mind:

1. When you pull out a common factor, you always pull out the one with the smallest exponent; do keep in mind that $-5/2$ is smaller than $1/2$.
2. Remember the following **additive principle**:

$$\text{(original exponent)} = \text{(exponent pulled out)} + \text{(exponent remaining inside)}$$

Now, let us turn to the problem:

$$\frac{3(1+x)^{1/3} - x(1+x)^{-2/3}}{(1+x)^{2/3}}$$

Pull out $(1+x)^{-2/3}$, keeping in mind that $1/3 = -2/3 + 1$ and $-2/3 = -2/3 + 0$:

$$\frac{(1+x)^{-2/3}[3(1+x)^1 - x(1+x)^0]}{(1+x)^{2/3}}$$

Anything raised to the 0th power is 1, hence

$$\frac{(1+x)^{-2/3}[3(1+x) - x]}{(1+x)^{2/3}}$$

We simplify the numerator as follows

$$\frac{(1+x)^{-2/3}(2x+3)}{(1+x)^{2/3}}$$

Relocate the factor with a negative exponent:

$$\frac{2x+3}{(1+x)^{2/3}(1+x)^{2/3}}$$

We simplify the numerator as follows

$$\frac{2x+3}{(1+x)^{4/3}}$$