Problem 1. This problem deals with the rates of change of a family of functions.

Part 1 (1). Let $f(t) = t^2$. Calculate the average rate of change of the function f between $t = x$ and $t = x + h$. (Make sure to simplify the fraction completely by cancelling out the extra h)

Solution. The average rate of change of the function f between $t = x$ and $t = x + h$ is defined as

$$\frac{f(x+h) - f(x)}{(x+h) - x}.$$

Thus, we have

$$\begin{aligned}\frac{f(x+h) - f(x)}{(x+h) - x} &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{(x+h)^2 - (x)^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h\end{aligned}$$

Part 2 (1). Plug in zero for h in the answer you produced in Part 1. What do you get?

Solution. We get $2x$.

Part 3 (1). The answer for Part 2 should be in terms of x; that is, it should be a function of x. This function, often labelled $f'(x)$, is called the *instantaneous rate of change* of the function f. Calculate the instantaneous rate of change of f at $x = 3$.

Solution. The instantaneous rate of change of f at $x = 3$ is $f'(3) = 2 \cdot 3 = 6$.

Part 4 (2). Now, let c be a constant, and let $g(t) = t^2 + c$ (Note: We have, in effect, shifted f vertically by c). Compute the instantaneous rate of change of the function g by repeating the processes outlined in Parts 1 and 2:

- Calculate the average rate of change of the function $g(t)$ between $t = x$ and $t = x + h$.
- Plug in zero for h.

Solution. The average rate of change of the function f between $t = x$ and $t = x + h$ is defined as

$$\frac{g(x+h) - g(x)}{(x+h) - x}.$$

Thus, we have

$$\begin{aligned}\frac{g(x+h)-g(x)}{(x+h)-x} &= \frac{g(x+h)-g(x)}{h} \\ &= \frac{[(x+h)^2+c]-[(x)^2+c]}{h} \\ &= \frac{x^2+2xh+h^2+c-x^2-c}{h} \\ &= \frac{2xh+h^2}{h} \\ &= 2x+h\end{aligned}$$

Now, if we plug in $h=0$, we get $2x$.

Part 5 (1). Compare the instantaneous rate of change of the function g with the instantaneous rate of change of the function f. What can you conclude about these two functions?

Solution. They are the same.

Part 6 (2). Let $c=5$, so that we have $g(x)=x^2+5$ (We are changing t to x, so as to revert back to the good-old xy-cooordinate). What is the vertex of the parabola?

Solution. g is already in the standard form. The vertex is $(0,5)$.

Part 7 (2). At which point x does the function g attain its minimum?

Solution. g attains its minimum at $x=0$.

Part 8 (1). Calculate the instantaneous rate of change of the function g at the point of minimum.

Solution. The instantaneous rate of change of g at $x=0$ is given by $g'(0)=0$.

Extra credit (2) What can you conclude about the relationship between the instantaneous rate of change of a function and its points of minima?

Solution. Whenever the function attains its minimum, the instantaneous rate of change is zero.