Notes on Set Theory

Set theory is not covered in the course, but some set-theoretic notions will be assumed throughout the course. This note aims to serve as an exposition and a handy reference for all such notions.

What is a Set?

A *set*, in mathematics, is simply a collection of objects with clear membership. That is, an object must either be in the set, or not in the set; "sort of in the set" is not acceptable. An object in a set is called an *element* of the set.

Let us consider a few examples.

Example 1. All students in Rutgers SAS Class of 2013

Example 1 is clearly a set, as it is clear who (or what) falls under this classification. Richard B. Freshman is in the set; Jeremiah the bullfrog is not.

Example 2. All female students in Rutgers SAS Class of 2013

Example 2 is also a set. Note that Example 2 describes a smaller set than Example 1 does. For one, Richard B. Freshman is not in the set anymore. Every element of the second set, however, does belong to the first set.

Example 2 is an example of a *subset.* The second set is a subset of the first set.

Example 3. All pretty students in Rutgers SAS Class of 2013

Example 3 is **not** a set. Though you may be convinced that you have a clear idea of who is pretty or not, it is just too vague.

Set Notations

There are two ways to write down a set: *roster* notation, and *set-builder* notation.

Roster notation is simple. We simply write down all elements of the set, and enclose them with curly brackets. The result is clearly a set, as we have indicated quite explicitly what belongs to the set.

Example 4. {Brower, Busch, Cooper, Neilson, Tillet}

Example 4 is an example of a set, written in roster notation; in fact, it is the set of all dining halls on Rutgers New Brunswick / Piscataway campus.

Example 5. {Neilson, Busch, Brower, Tillet, Cooper}

Example 5 is also the set of all dining halls on Rutgers New Brunswick / Piscataway campus. It would be odd to say that Example 4 and Example 5 denote two different sets, as we just gave them exactly the same label. In fact, in roster notation, *the order of the elements does not matter.*

Example 6. {Neilson, Neilson, Busch, Busch, Busch, Brower, Brower, Tillet, Cooper}

Likewise, Example 6 also denotes the same set. In roster notation, *duplicates only count once.*

Roster notation is clean and simple, but it can be quite burdensome as the set grows large. For one, writing Example 1 in roster notation would result in a very, very long list of names that we do not necessarily need. (Indeed, a *roster* of all freshmen!)

This is where set-builder notation comes in. Instead of writing down every single element, set-builder notation specifies a condition from which the set can be built.

Example 6. $\{x : x \text{ is a student in Rutgers SAS class of 2013}\}$

Example 6 is an example of a set, written in set-builder notation. You could treat the notation as a sentence: "a set of all x such that x is a student in Rutgers SAS class of 2013." The first x corresponds to "a set of all x." Note that changing the variable does not change the set:

Example 7. {☺ : ☺ is a student in Rutgers SAS class of 2013}

The colon is a shorthand for "such that." An equivalent shorthand is bar (|):

Example 8. $\{x | x \text{ is a student in Rutgers SAS class of 2013}\}$

The last part deserves special attention. It acts like a computer program, designed to do one job: examine each object—if it satisfies the specified condition, throw it in the set; if not, then move on. Logical connectives such as *and* and *or* can be used to write out the conditions:

Example 9. $\{x : x \text{ is a student } \textit{and } x \text{ is in Rutgers SAS class of 2013}\}$

I hope it is now clear to you that Example 1 and Examples 6-9 all denote the same set.

Let us conclude this section by considering a few mathematical examples; this is a math course, after all!

Example 10. $\{1, 2, 3, 4, 5\}$

Example 11. $\{x : 1 \leq x \leq 5 \textit{ and } x \text{ is an integer}\}$

Example 12. $\{x : 1 \leq x \leq 5\}$

Example 13. $\{x : 1 \leq x \leq 3 \textit{ or } 3 < x \leq 5\}$

Examples 10 and 11 denote the same set. Examples 12 and 13 denote the same set (can you see why?). Examples 11 and 12 do not denote the same set, since an "object" for set-builder notation is understood to be a real number, unless otherwise specified. For example, 1.5 and $\sqrt{2}$ are included in Example 12, but not in Example 11.

Set Operations

Between all mathematical objects exist operations. For example, we can add, subtract, multiply, or divide numbers, which result in another number. Likewise, there are operations between sets, aptly named *set operations*. There are quite a few, but we shall only consider two: *union* and *intersection*.

A union of two sets collect the elements in each set, and throw them into another set. If A and B are sets, we denote their union as $A \cup B$.

What do we mean by this? Simple: if x is an element of A, then x is an element of $A \cup B$; if x is an element of B, then x is an element of $A \cup B$.

Example 14. $\{1,2,3\} \cup \{4,5,6\} = \{1,2,3,4,5,6\}$

Example 14 is an example of the union operation. If two sets have duplicate elements, we only write them once in the resulting union:

Example 15. $\{1,2,3,4\} \cup \{3,4,5,6\} = \{1,2,3,4,5,6\}$

This omission makes sense, as we have discussed that $\{1,2,3,4,5,6\}$ and $\{1,2,3,4,3,4,5,6\}$ are the same set.

Another set operation we shall consider is the intersection operation. An intersection of two sets collect the elements that both sets share, and throw them into another set. That is, if x is an element of A *and* x is an element of B, then x is an element of $A \cap B$.

Example 16. $\{2,4,6\} \cap \{1,2,3,6\} = \{2.6\}$

From time to time, two sets may not share any element. Then, the resulting set is the *empty set*, which we denote as $\emptyset$.

Example 17. $\{1,3,5\} \cap \{2,4,6\} = \emptyset$.

The empty set does not have any element. Therefore, if you take the union of any set and the empty set, the resulting set is the original set, unchanged:

Example 18. $\{4,6,8,10\} \cup \emptyset = \{4,6,8,10\}$

If you take the intersection of any set and the empty set, however, the resulting set is $\emptyset$:

Example 19. $\{4,6,8,10\} \cap \emptyset = \emptyset$

Intervals

We finally come to the notion that we will be using again and again in this course: the intervals. An *interval*, intuitively, is a "chunk" of numbers.

Example 20. All real numbers between 1 and 4.

Example 20 is an example of an interval. Using set-roster notation, we can also write:

Example 21. $\{x : 1 < x < 4\}$

There are four types of intervals, each with its own *interval notation*:

Interval Notation	Set Notation
$[a, b]$	$\{x : a \leq x \leq b\}$
(a, b)	$\{x : a < x < b\}$
$[a, b)$	$\{x : a \leq x < b\}$
$(a, b]$	$\{x : a < x \leq b\}$

where a and b are real numbers such that $a < b$.

With this notation, Example 20 can be written as:

Example 22. $(1, 4)$

The notion of intervals can be extended to include unbounded set of numbers.

Example 23. All real numbers bigger than 4.

We also consider Example 23 is as an interval. We use the positive infinity symbol (∞) and the negative infinity symbol ($-\infty$) to denote unbounded intervals.

Example 24. $(4, \infty)$

A parenthesis is always used for ∞ and $-\infty$. An interval can never include ∞ or $-\infty$, for the simple reason that they are not numbers.

Since intervals are sets, the set operations can be applied to any two intervals. Sometimes, the operations produce another interval, as shown in Examples 25-28.

Example 25. $[1, 4] \cup [4, 7] = [1, 7]$

Example 26. $(1, 2) \cup (1.5, 8] = (1, 8]$

Example 27. $[-2, 2] \cap (-1, 3) = (-1, 2]$

Example 28. $(-\infty, \sqrt{2}) \cap (-4.5, \infty) = (-4.5, \sqrt{2})$

Some operations, however, do not produce a single interval:

Example 29. $(3, 6) \cup (-1, 2)$

Furthermore, taking the intersection of two intervals could produce an empty set:

Example 30. $(1, 2) \cap (2, 3) = \emptyset$

Exercises

Problem 1. Write out the following sets in roster notation:

1. $\{x : x < 5$ *and* x is a natural number$\}$
2. $\{1, 3, 7, 2, 5\} \cap \{9, 18, 25\}$
3. $\{x : x > 7$ *and* x is an integer$\} \cap \{x : x < 10$ *and* x is an integer$\}$
4. $\{5, 7, 9\} \cup \{0\}$
5. $[2, 5] \cap \{x : x$ is an integer$\}$
6. $(-2, 7) \cap \{x : x$ is a natural number$\}$

Problem 2. Write out the following sets in interval notation:

1. $\{x : x > 7$ *and* $x \leq 15\}$
2. $\{x : x < 9\}$
3. $[1, 5] \cap (2, 7)$
4. $(2, \infty) \cup (-\infty, 7)$
5. $(2, 7) \cup [7, 10)$
6. $[1, 5) \cup [2, 8)$