SUGGESTED SOLUTIONS FOR PROBLEM SET 1

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Remark. You can refer back to the definition of supremum (i.e., least upper bound) in the brief class notes. The problems here basically only use that definition and the axioms for the real numbers $\mathbb{R}$.

Exercise 1. Show directly from the definition that the supremum of a nonempty bounded subset S of $\mathbb{R}$ is unique. (In other words, if you suppose that g and h are both suprema of S, then you must show that $g = h$.)

Proof. Let $S \subseteq \mathbb{R}$ be bounded, and suppose that g and h are suprema of S. Since h is an upper bound of S, we have $g \leq h$. Likewise, g is an upper bound of S, so that $h \leq g$. It follows that $g = h$, whence the supremum is unique. □

Remark. We shall henceforth speak of *the* supremum, as opposed to *a* supremum.

Exercise 2. Let S be any nonempty subset of $\mathbb{R}$ which is bounded above. Define the new set $S + 7$ to be $\{x + 7 : x \in S\}$. Show that $S + 7$ is bounded above, and if $g = \sup(S)$, which we know exists by the least upper bound property, then show that $\sup(S + 7) = g + 7$.

Proof. We first observe that $g + 7$ is an upper bound: indeed, for any $s \in S + 7$, we have $s - 7 \in S$, so that $s - 7 \leq g$, or $s \leq g + 7$. Fix any upper bound t_0 of $S + 7$. Arguing as before, we see that $t_0 - 7$ is an upper bound of S, which implies that $g \leq t_0 - 7$, or $g + 7 \leq t_0$. It follows that $g + 7$ is the *least* upper bound of $S + 7$. □

Exercise 3. Let S be any nonempty subset of $\mathbb{R}$ which is bounded above. Define the new set $7S$ to be $\{7x : x \in S\}$. Show that $7S$ is bounded above, and if $g = \sup(S)$, which we know exists by the least upper bound property, then show that $\sup(7S) = 7g$.

Proof. We first observe that $7g$ is an upper bound: indeed, for any $s \in 7S$, we have $s/7 \in S$, so that $s/7 \leq g$, or $s \leq 7g$. Fix any upper bound t_0 of $7S$. Arguing as before, we see that $t_0/7$ is an upper bound of S, which implies that $g \leq t_0/7$, or $7g \leq t_0$. It follows that $7g$ is the *least* upper bound of $7S$. □

Exercise 4. Show that the supremum of any nonempty finite subset F of $\mathbb{R}$ is contained in F. (Note: a part of showing this property is showing, from the axioms, that F actually has a supremum, i.e., why is F bounded above?)

Proof. Let n be the cardinality of F, and write $F = \{x_1, x_2, \ldots, x_n\}$, where $x_i < x_j$ for all indices $1 \leq i < j \leq n$. The possibility of such a labelling is a consequence of the trichotomy principle. We observe that $x_n \geq x_i$ for each $1 \leq i \leq n$, whence F is bounded above. In particular, x_n is an upper bound of F. Since $x_n \in S$, any upper bound u of F must satisfy the inequality $x_n \leq u$. It thus follows that x_n is the supremum of F. □