SUGGESTED SOLUTIONS FOR PROBLEM SET 2

SPRING 2011, MATH 312:01

Remark. Note that the sentences in italics are comments; they are not things that need to be shown.

Exercise 1. As in the notes for class #2, we let O be a nonempty subset of $\mathbb{R}$, and for this homework exercise we will assume that O is bounded. We fix x in O, and set $m = \inf\{l : (l, x) \subset O\}$ and $M = \sup\{u : (x, u) \subset O\}$.

(These numbers are both finite in this case because of the boundedness assumption, and of course by construction we know that $x \in (m, M)$.)

Show that $(m, M) \subset O$.

Proof. Let us first investigate the degenerate cases. If $\{l : (l, x) \subset O\}$ is empty, then

$$\infty = \inf\{l : (l, x) \subseteq 0\},$$

so that the interval (m, M) is ill-defined. Likewise, if $\{u : (x, u) \subset O\}$ is empty, then

$$-\infty = \sup\{u : (x, u) \subset O\},$$

and the interval (m, M) is again ill-defined.

Please note that the openness assumption for O was supposed to be present: in our defense, we were not able to get an approval from the panda bears, and were subsequently sabotaged by a legion of pandas from posting a correction. If O is open, then there has to be an open neighborhood (a, b) of x contained in O, whence the above degenerate cases do not manifest. The existence of the maximal open intervals we are about to establish is used to prove that every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.

We may thus assume that both $\{l : (l, x) \subset O\}$ and $\{u : (x, u) \subset O\}$ are nonempty. Observe that

$$(m, x) = \bigcup_{m<l<x} (l, x),$$

which shows that $(m, x) \subseteq O$. To see this, we fix any $y_0 \in (m, x)$, and find $l_0 \in \mathbb{R}$ such that $m < l_0 < y_0$. Since $y_0 \in (l_0, x)$ and $(l_0, x) \subseteq O$, we see that $y_0 \in O$. Since y_0 was arbitrary, $(m, x) \subseteq O$.

Analogously, we can establish that $(x, M) \subseteq O$, for

$$(x, M) = \bigcup_{x<u<l} (x, u).$$

It follows that $(m, M) = (m, x) \cup \{x\} \cup (x, M)$ is contained in O, as desired. □

Exercise 2. Let O, x, and (m, M) be as in problem 1. If (a, b) is any interval satisfying $x \in (a, b) \subset O$, show that $(a, b) \subset (m, M)$.

(This is why we refer to (m, M) as being the "maximal" open subinterval containing x: it contains all the open subintervals which contain x.)

Proof. We *have* established this fact in the above proof; nevertheless, we reproduce the relevant portion:

Since

$$(m, x) = \bigcup_{m<l<x} (l, x) \quad \text{and} \quad (x, M) = \bigcup_{x<u<l} (x, u),$$

we see that $(a, x) \subseteq (m, x)$ and $(x, b) \subseteq (x, M)$. It follows that

$$(a, b) = (a, x) \cup \{x\} \cup (x, b) \subseteq (m, x) \cup \{x\} \cup (x, M) = (m, M),$$

as was to be shown. □

Exercise 3. Let $f(x)$ be continuous on the interval $[a, b]$, and let P be any partition

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Show that there is a marking Q of P such that the upper sum and the Riemann sum satisfy $U(P, f) = R(P, Q, f)$.

(You can use the known properties of continuous functions on closed bounded intervals from the earlier (pre-Chapter 5) portions of the text. Notice that the key is to figure out how to choose the points in Q. Also notice that if you can do this problem, you can easily do the analogous problem for the lower sum and corresponding (different Riemann sum.)

Proof. For each index $0 < i \leq n$, the subinterval $[x_{i-1}, x_i]$ is compact, whence f achieves a maximum on the subinterval. We let

$$\mathfrak{M}_i = \max\{f(x) : x \in [x_{i-1}, x_i]\},$$

and find $y_i \in [x_{i-1}, x_i]$ such that $f(y_i) = \mathfrak{M}_i$. Note that

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

equals $\mathfrak{M}_i$ for each i. This is a consequence of the following observation:

Lemma. *If x is the maximum of $S \subseteq \mathbb{R}$, then $x = \sup S$.*

Proof of the lemma. x is clearly an upper bound of S, whence $x \geq \sup S$. Since $x \in S$, we see that $x \leq \sup S$. □

We now let $Q = \{y_1, \ldots, y_n\}$, and observe that

$$R(P, Q, f) = \sum_{i=1}^{n} \mathfrak{M}_i (x_i - x_{i-1}) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = U(P, f),$$

which is the desired equality. □

Exercise 4. Let $f(x)$ be defined on the interval $[0, 1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2; \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

let P be <u>any</u> partition of the interval $0 = x_0 < x_1 < \ldots < x_n = 1$. Find $U(P, f)$ and $L(P, f)$ explicitly in terms of points in the partition.

(It is probably easiest to treat two cases separately: either $1/2$ is one of the partition points or it is not. Just give a name to the appropriately nearest points in the partition to $1/2$, and then you should be able to write the upper and lower sums in terms of those names.)

Proof. Let $P_1 = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$ containing $1/2$, and find k such that $x_k = 1/2$. Then

$$\begin{aligned}
L(P_1, f) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k} m_i(x_i - x_{i-1}) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k} 0 \cdot (x_i - x_{i-1}) + \sum_{i=k+1}^{n} 1 \cdot (x_i - x_{i-1}) \\
&= \sum_{i=k+1}^{n} x_i - x_{i-1} \\
&= x_n - x_k \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
U(P_1, f) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} M_i(x_i - x_{i-1}) + \sum_{i=k}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k} 0 \cdot (x_i - x_{i-1}) + \sum_{i=k}^{n} 1 \cdot (x_i - x_{i-1}) \\
&= \sum_{i=k}^{n} x_i - x_{i-1} \\
&= x_n - x_{k-1} \\
&= 1 - x_{k-1}.
\end{aligned}$$

□

We now consider a partition $P_2 = \{y_0, \ldots, y_m\}$ of $[0,1]$ not containing $1/2$, and find k such that

$$x_{k-1} < \frac{1}{2} < x_k.$$

On the interval $[x_{k-1}, x_k]$, we see that

$$\inf_{x \in [x_{k-1}, x_k]} f(x) = 0 \quad \text{and} \quad \sup_{x \in [x_{k-1}, x_k]} f(x) = 1.$$

Therefore,

$$\begin{aligned}
L(P_2, f) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k} m_i(x_i - x_{i-1}) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k} 0 \cdot (x_i - x_{i-1}) + \sum_{i=k+1}^{n} 1 \cdot (x_i - x_{i-1}) \\
&= \sum_{i=k+1}^{n} x_i - x_{i-1} \\
&= x_n - x_k \\
&= 1 - x_k
\end{aligned}$$

and

$$\begin{aligned}
U(P_2, f) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} M_i(x_i - x_{i-1}) + \sum_{i=k}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} 0 \cdot (x_i - x_{i-1}) + \sum_{i=k}^{n} 1 \cdot (x_i - x_{i-1}) \\
&= \sum_{i=k}^{n} x_i - x_{i-1} \\
&= x_n - x_{k-1} \\
&= 1 - x_{k-1}.
\end{aligned}$$

This completes the proof.