SUGGESTED SOLUTIONS FOR PROBLEM SET 3

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Remark. Once again, note that the sentences in italics are comments, they are not things that need to be shown.

Exercise 1. As in class, we let $O = \cup O_j$, with $O_1 = (1/3, 2/3)$; $O_2 = (1/9, 2/9) \cup (7/9, 8/9)$; $O_3 = (1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26/27)$; and so on, and we let C be the usual Cantor set obtained as the complement of O in the unit interval $[0, 1]$. Given any non-empty open interval (a, b) contained in $[0, 1]$, show that $(a, b) \cap O$ is non-empty.

(This statement is equivalent to saying that (a, b) is not entirely contained in C; it can also be phrased as the statement that O is dense in $[0, 1]$. As noted in class, think about the lengths of the longest remaining subintervals after the open sets O_j up to stage N have been removed, where N is chosen large enough depending on the width of (a, b).)

Proof. For each $n \in \mathbb{N}$, we set

$$C_n = [0, 1] \setminus \bigcup_{k=1}^{n} O_k.$$

By construction, the length of the longest continuous subinterval in C_n is $1/3^n$, whence C_n does not contain open intervals of length larger than $1/3^n$. Pick $N \in \mathbb{N}$ such that $1/3^N < b - a$. Then, for each $n \geq N$, C_n does not contain (a, b). It follows that

$$C = \bigcap_{n=1}^{\infty} C_n$$

cannot contain (a, b), as was to be shown. □

Exercise 2. Let O and C be as in Exercise 1, and as in class consider the characteristic function of C, namely, f defined on the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{for } x \in C; \\ 0 & \text{for } x \in O. \end{cases}$$

Show directly from the definition of continuity that f is continuous at every point of O, and discontinuous at every point of C.

(This function consequently has uncountably many discontinuities, and thus in class we have given an example of each of the following situations: a bounded function which has only countably many discontinuities and is Riemann integrable; a bounded function which has uncountably many discontinuities and is Riemann integrable; and a bounded function which has uncountably many discontinuities and is NOT Riemann integrable.)

Proof. Let $x \in [0, 1]$. If $x \in C$, then any neighborhood $(x - \delta, x + \delta)$ of x has a nonempty intersection with O, whence any $0 < \varepsilon < 1/2$ furnishes $y_0 \in (x - \delta, x + \delta) \cap O$ such that

$$|f(x) - f(y_0)| = |0 - 1| = 1 > \varepsilon.$$

Therefore, f is discontinuous at every point of C. If $x \in O$, we may find a neighborhood of $(x-\delta_0, x+\delta_0)$ entirely contained in O, via openness of O. Since f is identically 1 on O, it follows that any $y \in (x-\delta_0, x+\delta_0)$ satisfies

$$|f(x) - f(y)| = |1-1| = 0 < \varepsilon$$

for any $\varepsilon > 0$. It follows that f is continuous at every point of O. □

Exercise 3. Suppose $g : [a,b] \to \mathbb{R}$ is continuous except at $x_0 \in (a,b)$ and bounded. Prove that $g \in \mathfrak{R}(x)$ on $[a,b]$. See Exercises 24 and 25 for generalizations of this result.

Proof. Fix $\varepsilon > 0$, and let

$$M = \sup\{f(x) : x \in [a,b]\} \qquad \text{and} \qquad m = \inf\{f(x) : x \in [a,b]|\};$$

furthermore, we set $L = M - m$. g is continuous on $[a, x_0 - \varepsilon/6L]$, hence g is Riemann-integrable on $[a, x_0 - \varepsilon/6L]$, and there exists a partition P_1 of $[a, x_0 - \varepsilon/6L]$ such that $U(P_1, f) - L(P_1, f) < \varepsilon/3$. Likewise, g is continuous on $[x_0 + \varepsilon/6L, b]$, and we may find a partition P_2 of $[x_0 + \varepsilon/6L]$ such that $U(P_2, f) - L(P_2, f) < \varepsilon/3$.

We now set $P = P_1 \cup P_2$, and let

$$M' = \sup\{f(x) : x \in [x_0 - \varepsilon/6L, x_0 + \varepsilon/6L]\}$$

and

$$m' = \inf\{f(x) : x \in [x_0 - \varepsilon/6L, x_0 + \varepsilon/6L]\}.$$

Then

$$U(P,f) = U(P_1, f) + U(P_2, f) + M' \cdot \frac{\varepsilon}{3L}$$

and

$$L(P,f) = L(P_1, f) + U(P_2, f) + m' \cdot \frac{\varepsilon}{3L},$$

so that

$$\begin{aligned} U(P,f) - L(P,f) &= [U(P_1,f) - L(P_1,f)] + [U(P_2,f) - L(P_2,f)] + (M' - m') \cdot \frac{\varepsilon}{3L} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + L \cdot \frac{\varepsilon}{3L} \\ &= \varepsilon. \end{aligned}$$

Note that we have used the inequality $M' - m' \leq M - m = L$. It thus follows that f is Riemann-integrable on $[a,b]$. □

Exercise 4. Assume $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f\,dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

Proof. We suppose for a contradiction that $f(x_0) \neq 0$ for some $x_0 \in [a,b]$. We can then find a $\delta > 0$ such that $|f(x) - f(x_0)| < f(x_0)/2$, i.e., $f(x) > f(x_0)/2$, on $(x_0 - \delta, x_0 + \delta)$. Since $f(x) \geq 0$ for all $x \in [a,b]$, it follows that

$$U(P,f) > \frac{f(x_0)}{2} \cdot 2\delta = f(x_0)\delta$$

for each partition P, whence the upper integral of f cannot be 0. This is evidently absurd, for the value of $\int_a^b f\,dx$ was assumed to be zero. We thus conclude that $f(x) = 0$ for all $x \in [a,b]$. □