SUGGESTED SOLUTIONS FOR PROBLEM SET 4

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Exercise 1. Suppose $f \in \mathfrak{R}(x)$ on $[0,1]$. Define

$$a_n = \frac{1}{n} \sum_{n=1}^{\infty} f\left(\frac{k}{n}\right)$$

for all n. Prove $(a_n)_{n=1}^{\infty}$ converges to $\int_0^1 f\,dt$.

Proof. Observe that a_n equals the Riemann sum $S(P, f)$ with the partition

$$P = \{k/n : 0 \le k \le n\}$$

of mesh $1/n$, where each subinterval $[(k-1)/n, k/n]$ is marked by k/n. For each $\varepsilon > 0$, Theorem 5.6 furnishes a real number $\delta > 0$ and an integer $N > 1/\delta$ such that $n > N$ implies $|a_n - \int_0^1 f(t)\,dt| \le \varepsilon$. Since ε was arbitrary, we conclude that

$$\lim_{n\to\infty} a_n = \int_0^1 f(t)\,dt,$$

as was to be shown. □

Exercise 2. Suppose f is integrable on $[-b, b]$. Prove that

$$\int_{-b}^{b} f\,dx = 0$$

if f is odd, and

$$\int_{-b}^{b} f\,dx = 2\int_0^b f\,dx$$

if f is even.

Proof. For simplicity's sake, we shall freely use the theorems in chapter 5. By Theorem 5.10, $f \in \mathfrak{R}(x)$ on $[-b, 0]$, $f \in \mathfrak{R}(x)$ on $[0, b]$, and

$$\int_{-b}^{b} f(t)\,dt = \int_0^b f(t)\,dt + \int_{-b}^{0} f(t)\,dt.$$

Theorem 5.18, with the function $\phi : [0, b] \to \mathbb{R}$ defined by $\phi(x) = -x$, implies that

$$\int_{-b}^{0} f(t)\,dt = -\int_b^0 f(-t)\,dt.$$

If f is odd, then

$$-\int_b^0 f(-t)\,dt = \int_b^0 f(t)\,dt$$

whence the convention dictates that

$$\int_b^0 f(t)\,dt = -\int_0^b f(t)\,dt.$$

It now follows that

$$\int_{-b}^b f(t)\,dt = \int_0^b f(t)\,dt + \int_{-b}^0 f(t)\,dt = \int_0^b f(t)\,dt - \int_0^b f(t)\,dt = 0.$$

If f is odd, then

$$-\int_b^0 f(-t)\,dt = -\int_b^0 f(t)\,dt,$$

whence the convention dictates that

$$-\int_b^0 f(t)\,dt = \int_0^b f(t)\,dt.$$

It now follows that

$$\int_{-b}^b f(t)\,dt = \int_0^b f(t)\,dt + \int_{-b}^0 f(t)\,dt = 2\int_0^b f(t)\,dt,$$

as was to be shown. □

Remark. In accordance with the section in which this exercise is placed, we note that the statement can also be proven by observing that the Riemann sums of the symmetric partitions and markings of the two half-intervals are equal.

Exercise 3. Suppose f and g are differentiable on $[a, b]$ and f' and g' are integrable on $[a, b]$. Prove that $f'g$ and $g'f$ are integrable on $[a, b]$, and that

$$\int_a^b f'g\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f\,dx.$$

Proof. By product rule, we have $(fg)' = f'g + g'f$. f' and g' are differentiable, hence continuous. Therefore, $f'g$ and $g'f$ are continuous as well, which, in particular, implies that $(fg)'$ is integrable. Integrating both sides, we obtain

$$\int_a^b (fg)'\,dx = \int_a^b f'g\,dx + \int_a^b g'f\,dx.$$

The fundamental theorem of calculus implies

$$\int_a^b (fg)'\,dx = f(b)g(b) - f(a)g(a),$$

whence it follows from a simple algebraic manipulation that

$$\int_a^b f'g\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f\,dx.$$

This completes the proof. □

Exercise 4. Suppose $f \in \mathfrak{R}(x)$ on $[a, b]$ and $\frac{1}{f}$ is bounded on $[a, b]$. Prove that $\frac{1}{f} \in \mathfrak{R}(x)$ on $[a, b]$.

Proof. Since f and $1/f$ are bounded, we can find $C, M \in \mathbb{R}$ such that $|f| \leq C$, $|1/f| \leq M$, and $1/M < C$. Then $|f| \geq 1/M$, so that $1/M \leq |f| \leq C$. Therefore, we have

$$f([a,b]) \subseteq [-C, -1/M] \cup [1/M, C],$$

which is compact. Since $\phi(x) = 1/x$ is continuous on $[-C, -1/M] \cup [1/M, C]$, the desired conclusion follows from Theorem 5.11 $\square$