SUGGESTED SOLUTIONS FOR PROBLEM SET 5

SPRING 2011, MATH 312:01

Exercise 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is periodic and integrable on every closed interval. If p is the period of f, prove that for any $a \in \mathbb{R}$,

$$\int_0^p f\,dx = \int_a^{a+p} f\,dx.$$

Proof. Find $n \in \mathbb{Z}$ such that $a' = a - np$ and $0 \le a' < p$. Since f is p-periodic, a repeated application of the change-of-variables theorem and the "algebra of integrable functions" theorem yields:

$$\begin{aligned}\int_0^p f\,dx &= \int_0^{a'} f\,dx + \int_{a'}^p f\,dx = \int_{np}^{a'+np} f\,dx + \int_{a'+np}^{(n+1)p} f\,dx \\ &= \int_{np}^{a} f\,dx + \int_{a}^{(n+1)p} f\,dx = \int_{(n+1)p}^{p+a} f\,dx + \int_{a}^{(n+1)p} f\,dx \\ &= \int_a^{p+a} f\,dx,\end{aligned}$$

as was to be shown. □

Remark. In accordance with the section in which this exercise is placed, we note that the statement can also be proved by using the Riemann sums with corresponding partitions and markings of the two intervals, combined with the periodicity of f.

Exercise 2. Suppose $g : [a, b] \to \mathbb{R}$ is bounded and continuous except at $x_1, \ldots, x_n \in [a, b]$. Prove that $g \in \mathfrak{R}(x)$ on $[a, b]$.

Remark. Let us discuss the main ideas of the following (rather long) proof before indulging ourselves in the nitty-gritty technical details. We have seen that the key to establish the Riemann-integrability of a continuous function is that the function is *uniformly continuous* on closed intervals: that is, given a small enough subinterval (say of length smaller than δ), the total variation of the function is very small (within ε, for example).

When there are "a small enough number" of points of discontinuity, then some (but not all) subintervals will contain those "bad points," where the function is no longer uniformly continuous. Therefore, we cannot contain the total variation of the function in a small range on those subintervals. Since the function is assumed to be bounded, however, the total variation is still within some finite number. If we can bound the total length of the "bad" subintervals with a small enough number, then the product of the "variation on the y axis" and the "variation on the x axis"—which is what the "difference of the upper sum and the lower sum" is, essentially—will still be small.

Proof. Fix $\varepsilon > 0$, and find an $M > 0$ such that $|g(x)| < M$ for all $x \in [a,b]$. If $\varepsilon \geq 2M(b-a)$, then finding a partition P such that $U(P,f) - L(P,f) < \varepsilon$ is trivial. Therefore, we may well assume that $\varepsilon < 2M(b-a)$.

For each $1 \leq i \leq n$ we define an open interval $I_i = (x_i - \varepsilon/8Mn, x_i + \varepsilon/8Mn)$, and let

$$X = [a,b] \setminus \bigcup_{i=1}^{n} I_i = [a,b] \cap \left(\bigcup_{i=1}^{n} I_i \right)^c .$$

$[a,b]$ is closed, and each I_i is open, hence X is closed. Furthermore, $X \subseteq [a,b]$, so that X is compact. Note also that X is nonempty, for the sum of the *length* of each I_i (namely, $(\varepsilon/4Mn) \cdot n = \varepsilon/4M$) is less than $b-a$.

g is continuous on the compact set X, hence g is uniformly continuous, and we may find $\delta > 0$ such that $|x-y| < \delta$ implies $|g(x) - g(y)| < \varepsilon/2(b-a)$ for all $x, y \in X$. Now, let $P = \{p_0, \ldots, p_m\}$ be any partition of $[a,b]$ satisfying the following properties:

(1) The mesh of P is smaller than δ;
(2) P must contain both endpoints of each open interval I_i.

Take J to be the set of indices $\{1, \ldots, n\}$. We shall divide J into two parts as follows: if the subinterval $[p_{j-1}, p_j]$ is entirely contained in X, then $j \in J_g$; otherwise, $j \in J_b$.

We are now ready to make our estimate. We first observe that

$$U(P,f) - L(P,f) = \sum_{j=1}^{m} (M_j - m_j)(p_j - p_{j-1}) = \sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) + \sum_{k \in J_b} (M_k - m_k)(p_k - p_{k-1}).$$

Since $\mu(P) < \delta$, the "total variation" of f on each subinterval $[p_{j-1}, p_j] \subseteq X$ is within $\varepsilon/2(b-a)$. More specifically, condition (1) stipulates that f is uniformly continuous on the intevral $[p_{j-1}, p_j]$ of length less than δ, hence any $x, y \in [p_{j-1}, p_j]$ yields $|g(x) - g(y)| < \varepsilon$. Finally, $[p_{j-1}, p_j]$ is compact, whence we can find points $c_j, d_j \in [p_{j-1}, p_j]$ such that $g(c_j) = M_j$ and $g(d_j) = m_j$. We may thus conclude that

$$|M_j - m_j| = |g(c_j) - g(d_j)| < \varepsilon/2(b-a),$$

which immediately yields the inequality

$$\sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) < \sum_{j \in J_g} \frac{\varepsilon}{2(b-a)} (p_j - p_{j-1}) = \frac{\varepsilon}{2(b-a)} \sum_{j \in J_g} p_j - p_{j-1}.$$

$J_g \subseteq \{1, \ldots, n\}$, and each $p_j - p_{j-1}$ is postive, we have the inequality

$$\sum_{j \in J_g} p_j - p_{j-1} \leq \sum_{j=1}^{n} p_j - p_{j-1} = b - a,$$

and it follows that

$$\sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) < \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \frac{\varepsilon}{2}.$$

Now, condition (2) bounds the "total length" of the remaining subintervals. Indeed, (2) implies that each subinterval furnished by P is either in X or in $[a,b] \setminus X$. It then follows that the union of the subintervals corresponding to the indices in J_b is precisely $[a,b] \setminus X$, i.e.,

$$[a,b] \setminus X = \bigcup_{j \in J_b} [p_{j-1}, p_j].$$

Since $[a,b] \setminus X$ is the union of n intervals of length at most $\varepsilon/4Mn$, we have the following inequality:

$$\sum_{j \in J_b} p_j - p_{j-1} \leq n \cdot \frac{\varepsilon}{4Mn} = \frac{\varepsilon}{4M}.$$

In addition, g is bounded by M on $[a,b]$, so that $|g(x) - g(y)| < 2M$ for any $x, y \in [a,b]$. Therefore,

$$\sum_{j \in J_b} (M_i - m_i)(p_j - p_{j-1}) < 2M \sum_{j \in J_b} p_j - p_{j-1} \leq 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

It now follows that

$$U(P,f) - L(P,f) = \sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) + \sum_{k \in J_b} (M_k - m_k)(p_k - p_{k-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whence $g \in \mathfrak{R}(x)$ on $[a,b]$, as was to be shown. □

Exercise 3. Suppose f and g are integrable on $[a,b]$. Define $h(x) = \max\{f(x), g(x)\}$. prove that h is integrable on $[a,b]$.

Proof. Observe that

$$h(x) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

for all $x \in [a,b]$. The desired result now follows from Theorem 5.12 and Theorem 5.9(i). □

Remark. Analogously we have the following identity for any $a, b \in \mathbb{R}$:

$$\min\{a,b\} = \frac{a+b}{2} - \frac{|a-b|}{2}.$$

Exercise 4. Assume $f : [a,b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for all $x \in [a,b]$, and

$$M = \sup\{f(x) : x \in [a,b]\}.$$

Show that

$$\left(\left[\int_a^b [f(x)]^n \, dx \right]^{1/n} \right)_{n=1}^{\infty}$$

converges to M.

Proof. If f is identically zero, then the equality holds trivially. We therefore assume that $M > 0$. f is a continuous function on a compact interval $[a,b]$, hence there exists $m \in [a,b]$ such that $f(m) = M$. We fix $\varepsilon > 0$, and pick a neighborhood (c,d) of m in $[a,b]$ such that $|f(x) - M| < \varepsilon$ for all $x \in (c,d)$. Observe that

$$(M - \varepsilon)^n (d - c) = \int_c^d (M - \varepsilon)^n \, dx \leq \int_c^d [f(x)]^n \, dx \leq \int_a^b [f(x)]^n \, dx \leq \int_a^b M^n \, dx = M^n (b - a).$$

The inequality we shall need is

$$(M - \varepsilon)^n (d - c) \leq \int_a^b [f(x)]^n \, dx \leq M^n (b - a).$$

Taking the nth root, we obtain

$$(M - \varepsilon)(d - c)^{1/n} \leq \left(\int_a^b [f(x)]^n \, dx \right)^{1/n} \leq M(b - a)^{1/n}.$$

As $n \to \infty$, both $(d-c)^{1/n}$ and $(b-a)^{1/n}$ tend to 1, so that

$$(M-\varepsilon) \le \lim_{n\to\infty} \left(\int_a^b [f(x)]^n \, dx \right)^{1/n} \le M.$$

Since ε was arbitrary, we conclude that

$$\lim_{n\to\infty} \left(\int_a^b [f(x)]^n \, dx \right)^{1/n} = M,$$

as was to be shown. □