SUGGESTED SOLUTIONS FOR PROBLEM SET 6

SPRING 2011, MATH 312:01

Exercise 1. Let O_1 and O_2 be disjoint bounded open subsets of $\mathbb{R}$. Show directly from the definition of the measure of open set that $m(O_1 \cup O_2) = m(O_1) + m(O_2)$.

(This case is much simpler than the proposition considered in class, where the sets were allowed to overlap, because of the key part of the definition of the measure of an open set in terms of the widths of its disjoint maximal open subintervals.)

Proof. Let $\{E_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ be collections of pairwise-disjoint open intervals such that

$$O_1 = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad O_2 = \bigcup_{n=1}^{\infty} F_n.$$

Since O_1 and O_2 are disjoint, $\{E_n\}_{n\in\mathbb{N}} \cup \{F_n\}_{n\in\mathbb{N}}$ is a collection of pairwise-disjoint open intervals whose union is $O_1 \cup O_2$. It now suffices to note that

$$m(O_1 \cup O_2) = \sum_{n=1}^{\infty} m(E_n) + \sum_{n=1}^{\infty} m(F_n) = m(O_1) + m(O_2),$$

as was to be shown. □

Exercise 2. Let E be a bounded subset of $\mathbb{R}$ with $m(E) = 0$. Fix any real number a and any positive number b. Show that $m(E + a) = 0$ and $m(bE) = 0$.

(Note that we have defined these notation before; $E + a$ is the set of all points of the form $x + a$ for x in E, and bE is the set of all points of the form bx for x in E.)

Proof. If $b = 0$, then $bE = \{0\}$, which is clearly of measure zero. We therefore assume that $b \neq 0$. Fix $\varepsilon > 0$, and let $\{O_n\}_{n\in\mathbb{N}}$ be a collection of open intervals such that

$$E \subseteq \bigcup_{n=1}^{\infty} O_n \quad \text{and} \quad \sum_{n=1}^{\infty} m(O_n) < \varepsilon \cdot \min\left\{b, \frac{1}{b}\right\}.$$

Then $\{O_n + a\}_{n=1}^{\infty}$ is a cover of $E + a$, and $\{bO_n\}_{n=1}^{\infty}$ a cover of bE, both of which have the total length of at most ε. The desired result follows. □

Exercise 3. Consider the half-open interval $E = [0, 1)$. For any $\varepsilon > 0$, produce a compact set K_ε and an open set O_ε such that $K_\varepsilon \subseteq E \subseteq O_\varepsilon$ and $m(O_\varepsilon \setminus K_\varepsilon) < \varepsilon$.

(Note that the set $O_\varepsilon \setminus K_\varepsilon$ is itself open, so that its measure has been defined.)

Proof. Fix $\varepsilon > 0$, and set $\eta = \min\{1, \varepsilon\}$. Let $K_\varepsilon = [\eta/8, 1 - \eta/8]$ and $O_\varepsilon = (-\eta/8, 1 + \eta/8)$. Then $K_\varepsilon \subseteq E \subseteq O_\varepsilon$, and

$$m(O_\varepsilon \setminus K_\varepsilon) = m((-\eta/8, \eta/8) \cup (1-\eta/8, 1+\eta/8)) = m((-\eta/8, \eta/8)) + m((1-\eta/8, 1+\eta/8)) = \frac{\eta}{2} < \varepsilon,$$

as was to be shown. □

Exercise 4. Let

$$E = (1/2, 1] \cup (1/8, 1/4] \cup (1/32, 1/16] \cup \cdots = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}} \right].$$

For any $\varepsilon > 0$, produce a compact set K_ε and an open set O_ε such that $K_\varepsilon \subseteq E \subseteq O_\varepsilon$ and $m(O_\varepsilon \setminus K_\varepsilon) < \varepsilon$.

(Note that there is a technical point: you might try just looking at the infinite collection of open sets and compact sets that mimic on a smaller scale what you did in problem 3. But the union of an infinite collection of compact sets is NOT necessarily compact. So you have to stop with finitely many. So...)

Proof. Fix $\varepsilon > 0$, and let $\eta = \min\{\varepsilon, 1/8\}$. Find $N \in \mathbb{N}$ such that

$$\frac{1}{3 \cdot 2^{2N+1}} < \frac{\eta}{2}.$$

We now set

$$O_\varepsilon = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{2n+1}} - \frac{\eta}{2^{n+3}}, \frac{1}{2^{2n}} + \frac{\eta}{2^{n+3}} \right) \quad \text{and} \quad K_\varepsilon = \bigcup_{n=1}^{N} \left[\frac{1}{2^{2n+1}} + \frac{\eta}{2^{n+3}}, \frac{1}{2^{2n}} - \frac{\eta}{2^{n+3}} \right].$$

Then K_ε is compact, O_ε is open, $K_\varepsilon \subseteq E \subseteq O_\varepsilon$, and

$$\begin{aligned}
& m(O_\varepsilon \setminus K_\varepsilon) \\
= \; & m\left(\left(\bigcup_{n=1}^{N} \left(\frac{1}{2^{2n+1}} - \frac{\eta}{2^{n+3}}, \frac{1}{2^{2n+1}} + \frac{\eta}{2^{n+3}} \right) \right) \cup \left(\bigcup_{n=1}^{N} \left(\frac{1}{2^{2n}} - \frac{\eta}{2^{n+3}}, \frac{1}{2^{2n}} + \frac{\eta}{2^{n+3}} \right) \right) \right. \\
& \left. \cup \bigcup_{n=N+1}^{\infty} \left(\frac{1}{2^{2n+1}} - \frac{\eta}{2^{n+3}}, \frac{1}{2^{2n}} + \frac{\eta}{2^{n+3}} \right) \right) \\
= \; & \sum_{n=1}^{N} \frac{\eta}{2^{n+2}} + \sum_{n=1}^{N} \frac{\eta}{2^{n+2}} + \sum_{n=N+1}^{\infty} \left(\frac{1}{2^{2n+1}} + \frac{\eta}{2^{n+2}} \right) \\
= \; & \frac{(2^N - 1)\eta}{2^{N+2}} + \frac{(2^N - 1)\eta}{2^{N+2}} + \frac{1}{3 \cdot 2^{2N+1}} + \frac{\eta}{2^{N+2}} \\
= \; & \frac{(2^N - 1 + 2^N - 1 + 1)\eta}{2^{N+2}} + \frac{1}{3 \cdot 2^{2N+1}} \\
= \; & \frac{(2^{N+1} - 1)\eta}{2^{N+2}} + \frac{1}{3 \cdot 2^{2N+1}} \\
= \; & \frac{\eta}{2} - \frac{\eta}{2^{N+2}} + \frac{1}{3 \cdot 2^{2N+1}} \\
< \; & \frac{\eta}{2} + \frac{1}{3 \cdot 2^{2N+1}} \\
< \; & \frac{\eta}{2} + \frac{\eta}{2} \\
= \; & \eta \\
\leq \; & \varepsilon,
\end{aligned}$$

as was to be shown. □