SUGGESTED SOLUTIONS FOR PROBLEM SET 7

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Exercise 1. Let E be a bounded measurable subset of $\mathbb{R}$, and let $q \notin E$ be a single point. Show that $F = E \cup \{q\}$ is measurable directly from the definition of measurability, by showing that for any $\varepsilon > 0$ there is a compact set K_ε and and open set O_ε such that $K_\varepsilon \subseteq F \subseteq O_\varepsilon$ and $m(O_\varepsilon \setminus K_\varepsilon) < \varepsilon$.

Note that we could use the Lemma about the union of two measurable sets to prove this result, but I am asking you to go straight back to the definition rather than using the Lemma.

Proof. Fix $\varepsilon > 0$, and find an open superset $O_1 \supseteq E$ and a compact subset $K \subseteq E$ such that

$$m(O_1 \setminus K) < \frac{\varepsilon}{2}.$$

If O_1 contains q, then we are done. Otherwise, we let $O_2 = (q - \varepsilon/4, q + \varepsilon/4)$, and set $O = O_1 \cup O_2$. Then O is open, and $O \setminus K \subseteq O_1 \setminus K \cup O_2$, whence by monotonicity

$$m(O \setminus K) \leq m(O_1 \setminus K \cup O_2).$$

It now follows from subadditivity that

$$m(O_1 \setminus K \cup O_2) \leq m(O_1 \setminus K) + m(O_2),$$

and so

$$m(O \setminus K) \leq m(O_1 \setminus K) + m(O_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as was to be shown. □

Exercise 2. Let E be a nonempty bounded measurable subset of $\mathbb{R}$, and let $p \in E$ be a single point. Show that $G = E \setminus \{p\}$ is measurable directly from the definition of measurability, by showing that for any $\varepsilon > 0$ there is a compact set K_ε and and open set O_ε such that $K_\varepsilon \subseteq E \subseteq O_\varepsilon$ and $m(O_\varepsilon \setminus K_\varepsilon) < \varepsilon$.

This result also could be deduced from a combination of the Lemmas already established, but again you should be able to establish it directly from the definition.

Proof. Fix $\varepsilon > 0$, and find an open superset $O \supseteq E$ and a compact subset $K_1 \supseteq E$ such that

$$m(O \setminus K_1) < \frac{\varepsilon}{2}.$$

If K_1 does not contain p, then we are done. Otherwise, we let $K = K_1 \setminus (p - \varepsilon/4, p + \varepsilon/4)$. Then K is compact, and $O \setminus K \subseteq O \setminus K_1 \cup (p - \varepsilon/4, p + \varepsilon/4)$, whence by monotonicity

$$m(O \setminus K) \leq m(O \setminus K_1 \cup (p - \varepsilon/4, p + \varepsilon/4)).$$

It now follows from subadditivity that

$$m(O \setminus K_1 \cup (p - \varepsilon/4, p + \varepsilon/4)) \leq m(O \setminus K_1) + m((p - \varepsilon/4, p + \varepsilon/4)),$$

and so

$$m(O \setminus K) \leq m(O \setminus K_1) + m((p - \varepsilon/4, p + \varepsilon/4)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as was to be shown. $\square$

Exercise 3. Let E be a bounded measurable set. Show that there is a sequence of compact sets $(K_n)_{n=1}^{\infty}$ contained in E and a set of measure zero N such that, for $K = \bigcup K_n$, we have $E = F \cup N$.

Note that we can find open sets and compact sets $K_n \subseteq E \subseteq O_n$ with $m(O_n \setminus K_n) < 1/n$. For F the union of these compact sets, can you show that $E \setminus F$ has measure zero?

Proof. Since E is measurable, each $n \in \mathbb{N}$ admits an open superset $O_n \supseteq E$ and a compact subseteq $K_n \subseteq E$ such that

$$m(O_n \setminus K_n) < \frac{1}{n}.$$

Let $F = \bigcup K_n$. For each $n \in \mathbb{N}$, we have

$$E \setminus F \subseteq O_n \setminus F \subseteq O_n \setminus K_n,$$

whence $m(E \setminus F) \leq m(O_n \setminus K_n) < 1/n$ by monotonicity. It follows that

$$m(E \setminus F) < \frac{1}{n}$$

for all $n \in \mathbb{N}$, and so $m(E \setminus F) = 0$. Setting $N = E \setminus F$, we see that $E = F \cup N$. $\square$

Exercise 4. Let E be a bounded measurable set. Show that there is a sequence of bounded open sets O_n containing E and a set of measure zero M such that, for $G = \bigcap O_n$, we have $E = G \setminus M$.

Proof. Since E is measurable, a bounded open superset $O_n \supseteq E$ and a compact subseteq $K_n \subseteq E$ such that

$$m(O_n \setminus K_n) < \frac{1}{n}.$$

Let $G = \bigcap O_n$. For each $n \in \mathbb{N}$, we have

$$G \setminus E \subseteq O_n \setminus E \subseteq O_n \setminus K_n,$$

whence $m(G \setminus E) \leq m(O_n \setminus K_n) < 1/n$ by monotonicity. It follows that

$$m(G \setminus E) < \frac{1}{n}$$

for all $n \in \mathbb{N}$, and so $m(G \setminus E) = 0$. Setting $M = G \setminus E$, we see that $E = G \setminus M$. $\square$