SUGGESTED SOLUTIONS FOR PROBLEM SET 8

SPRING 2011, MATH 312:01

Exercise 1. Consider the general compact interval $[a, b]$. Show *directly from the definition of measure* that $m([b, a]) = b - a$. This means that you must show that for any open set O containing $[a, b]$, you have to note why $b - a \leq m(O)$, and that for any $\varepsilon > 0$, you must produce an open set O_ε containing $[a, b]$ such that $m(O_\varepsilon) \leq b - a + \varepsilon$.

Proof. For any open superset $O \supseteq [a, b]$, we have $(a, b) \subseteq [a, b] \subseteq O$, whence by monotonicity

$$b - a = m((a, b)) \leq m(O).$$

Given any $\varepsilon > 0$, the open interval $O_\varepsilon = (a - \varepsilon/4, b + \varepsilon/4)$ is an open superset of $[a, b]$ such that

$$m(O_\varepsilon) = b - a + \frac{\varepsilon}{2} \leq b - a + \frac{\varepsilon}{2}.$$

Therefore,

$$b - a \leq m([a, b]) \leq b - a + \varepsilon$$

for all $\varepsilon > 0$, whence $m([a, b]) = 0$. □

Exercise 2. Consider the general bounded half-open, half-closed interval $[a, b)$. Show *directly from the definition of measure* that $m([a, b)) = b - a$. This means that you must show that for any open set O containing $[a, b)$, you have to note why $b - a \leq m(O)$, and that for any open set O containing $[a, b)$, you must produce an open set O_ε containing $[a, b)$ such that $m(O_\varepsilon) \leq b - a + \varepsilon$.

Proof. For any open superset $O \supseteq [a, b)$, we have $(a, b) \subseteq [a, b) \subseteq O$, whence by monotonicity

$$b - a = m((a, b)) \leq m(O).$$

Given any $\varepsilon > 0$, the open interval $O_\varepsilon = (a - \varepsilon/4, b + \varepsilon/4)$ is an open superset of $[a, b)$ such that

$$m(O_\varepsilon) = b - a + \frac{\varepsilon}{2} \leq b - a + \frac{\varepsilon}{2}.$$

Therefore,

$$b - a \leq m([a, b)) \leq b - a + \varepsilon$$

for all $\varepsilon > 0$, whence $m([a, b]) = 0$. □

Note that, since we already know that the measure of the open interval (a, b) has measure $b - a$, problems 1 and 2 (allowing for the symmetric version of problem 2 with the left end open and right end closed) then mean that we now know the Lebesgue measure of any interval, hence of any disjoint union of intervals.

Exercise 3. This is a two-part problem.

(1) Let E_n be an increasing sequence of bounded measurable subsets of $\mathbb{R}$, which means
$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots .$$
Let $E = \bigcup E_n$, and suppose that E is bounded. Show that $m(E_n) \to m(E)$ as $n \to \infty$.

(2) Let D_n be a decreasing sequence of bounded measurable subseteq of $\mathbb{R}$, which means
$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots .$$
Let $D = \bigcap D_n$. Show that $m(D_n) \to m(D)$ as $n \to \infty$.

Remark. The first result is known as *continuity from below*, and the second *continuity from above.*

Proof. Define a sequence $(G_n)_{n=1}^{\infty}$ by setting $G_1 = E_1$ and $G_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then $(G_n)_{n=1}^{\infty}$ is a sequence of pairwise-disjoint measurable sets whose union is E. It follows from countable additivity that
$$m(E) = \sum_{n=1}^{\infty} m(G_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(G_n) = \lim_{N\to\infty} m\left(\bigcup_{n=1}^{N} G_n\right) = \lim_{N\to\infty} m(E_N),$$
as was to be shown.

Similarly, we can define a sequence $(F_n)_{n=1}^{\infty}$ by setting $F_n = D_n \setminus D_{n+1}$ for each $n \in \mathbb{N}$. Then
$$D_1 = D \cup \left(\bigcup_{n=1}^{\infty} F_n\right)$$
is a disjoint union of measurable sets, whence by countable additivity
$$\begin{aligned} m(D_1) &= m(D) + \sum_{n=1}^{\infty} m(F_n) \\ &= m(D) + \lim_{N\to\infty} \sum_{n=1}^{N-1} m(F_n) \\ &= m(D) + \lim_{N\to\infty} \sum_{n=1}^{N-1} m(D_n) - m(D_{n+1}) \\ &= m(D) + \lim_{N\to\infty} m(D_1) - m(D_N) \\ &= m(D) + m(D_1) - \lim_{N\to\infty} m(D_N). \end{aligned}$$
It now follows that
$$m(D) = \lim_{N\to\infty} m(D_N),$$
as was to be shown. □

Exercise 4. Let $\mathcal{S}$ be the set of all measurable subsets E of $[0,1]$ such that $m(E) = 0$ or $m(E) = 1$. Show that $\mathcal{S}$ is a σ-algebra of subsets of $[0,1]$, i.e., $\mathcal{S}$ contains $[0,1]$ itself; if $E \in \mathcal{S}$, then $[0,1] \setminus E \in \mathcal{S}$; and the union of any countable collection of elements of $\mathcal{S}$ is in $\mathcal{S}$.

Note that Lebesgue measure is then a measure on this σ-algebra. This might look like a rather odd σ-algebra, but it is very important in probability theory.

Remark. Note that if $m(E) = 1$, then $m([0,1] \smallsetminus E) = 0$, and so E is the complement of a measure-zero set. Measure-zero sets are commonly referred to as *null sets*, and their complements *co-null sets*. $\mathcal{S}$ is then an example of σ-algebra of null and co-null sets. A property that holds on a co-null set is said to be satified *almost everywhere*.

Proof. The whole set $[0,1]$ is evidently in $\mathcal{S}$, for $m([0,1]) = 1$. Given $E \in \mathcal{S}$, we let $F = [0,1] \smallsetminus E$. By countable additivity, $m(E) + m(F) = m([0,1]) = 1$, and so $m(F) = 0$ if $m(E) = 1$, and $m(F) = 1$ if $m(E) = 0$. In both cases, $F \in \mathcal{S}$. We now suppose that $(E_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{S}$, and let E be the union. If $m(E_N) = 1$ for some $N \in \mathcal{N}$, then monotonicity implies that

$$1 = m(E_N) \leq m(E) \leq m([0,1]) = 1,$$

whence $m(E) = 1$. In this case, $E \in \mathcal{S}$. If not, then $m(E_n) = 0$ for all $n \in \mathbb{N}$. For each $\varepsilon > 0$ and $n \in \mathbb{N}$, we can then find an open superset $O(\varepsilon, n) \supseteq E_n$ such that $m(O(\varepsilon, n)) < \varepsilon / 2^n$. Then

$$m(E) \leq \sum_{n=1}^{\infty} m(O(\varepsilon, n)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

by subadditivity. Since $\varepsilon > 0$ was arbitrary, it follows that $m(E) = 0$, and so $E \in \mathcal{S}$. This completes the proof. $\square$