SUGGESTED SOLUTIONS FOR PROBLEM SET 9

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Exercise 1. Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be measurable functions and let E be a measurable set subset of $[a,b]$. Set

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in [a,b] \setminus E \\ f_2(x) & \text{for } x \in E. \end{cases}$$

Show that f is measurable.

A much simpler proof, due to Mr. Calkins:

Proof. Recall that the collection of Lebesgue-measurable sets is a σ-algebra, viz., is closed under countable unions, countable intersections, and complements. We now observe that

$$f^{-1}(O) = \big(f_1^{-1}(O) \cap ([a,b] \setminus E)\big) \cup \big(f_2^{-1}(O) \cap E\big)$$

for any open set O, whence f is measurable. □

Exercise 2. Let $f : [a,b] \to \mathbb{R}$ be a measurable function. Let $N = \{x : f(x) = 0\}$. Define $1/f : E \to \mathbb{R}$ as follows:

$$(1/f)(x) = \begin{cases} \frac{1}{f(x)} & \text{for } x \in [a,b] \setminus N \\ 17 & \text{for } x \in N. \end{cases}$$

Proof. Since $g_1(x) = 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$, the composition $(g_1 \circ f)(x) = 1/f(x)$ is measurable on $[a,b] \setminus N$. The constant function $g_2(x) = 17$ is measurable on $[a,b]$, as

$$g_2(O) = \begin{cases} [a,b] & \text{if } 17 \in O; \\ \varnothing & \text{if } 17 \notin O; \end{cases}$$

for any set $O \subseteq \mathbb{R}$. Since f is measurable, $N = f^{-1}(\{0\})$ is measurable as well. It now follows from Exercise 1 that $1/f$ is measurable on $[a,b]$, as was to be shown. □

Exercise 3. We know that if $(f_n)_{n=1}^{\infty}$ are functions defined on $[a,b]$ that are uniformly bounded, and $f_n \to f$, then $\int f_n \to \int f$. But the result does *not* in general hold in the other direction, in spectacular fashion. That is to say, just because the integrals of a sequence of measurable functions converge, it does not mean that the functions themselves converge anywhere. Find a sequence of measurable functions f_n that are all defined on $[0,1]$ and are the indicator functions of measurable sets E_n (i.e. find a collection of measurable subsets E_n of $[0,1]$ and let the function f_n be 1 on E_n and 0 otherwise) such that $m(E_n) \to 0$ (i.e. $\int_{[0,1]} f_n \to 0$), but such that $f_n(x)$ does not converge for any $x \in [0,1]$.

Proof. Let $E_0 = [0,1]$, $E_1 = [0,1/2]$, $E_2 = [1/2,1]$, $E_3 = [0,1/3]$, $E_4 = [1/3,2/3]$, $E_5 = [2/3,1]$, $E_6 = [0,1/4]$, and, in general,

$$E_n = \begin{cases} [0,1/n] & \text{if } n = \displaystyle\sum_{i=0}^{N} i = \frac{i(i+1)}{2} \text{ for some integer } N \geq 0; \\ [N/n, N+1/n] & \text{if } n - N = \displaystyle\sum_{i=0}^{N} i = \frac{i(i+1)}{2} \text{ for some integer } N \geq 0; \end{cases}$$

so that $m(E_n) \to 0$. If we let f_n be the indicator function on E_n, then $\int f_n = m(E_n) \to 0$. The sequence $(f_n)_{n=1}^{\infty}$ does not converge anywhere, nevertheless. To see this, we fix a point x in $[0,1]$ and a natural number n. If we let $N = 0 + 1 + 2 + \cdots + n$, then the sets $E_N, \cdots, E_{N+n}$ partition $[0,1]$. Therefore, there exists a natural number N' between N and $N+n$ such that $x \in E_{N'}$, and that $x \notin E_{N'+1}$. It follows that

$$|f_{N'}(x) - f_{N'+1}(x)| = 1 > \frac{1}{2},$$

whence $(f_n)_{n=0}^{\infty}$ is nowhere Cauchy, and so nowhere convergent. □

Exercise 4. Given $(f_n)_{n=1}^{\infty}$ defined on $[a,b]$ and measurable and converging on $[a,b]$, $f_n \to f$, we know that, e.g., the hypothesis of uniform boundedness implies that $\int f_n \to \int f$. In order to see why *some* hypothesis (e.g. uniform boundedness) is necessary, consider the functions g_n defined on $[0,1]$ by

$$g_n(x) = \begin{cases} 2^n & \text{if } 1/2^n \leq x \leq 1/2^{n-1}; \\ 0 & \text{otherwise.} \end{cases}$$

Explain why each function g_n is measurable, and $g_n(x) \to 0$ for *each* x in $[0,1]$, but $\int_{[0,1]}$ does *not* converge to $0 = \int_{[0,1]} 0$.

Proof. For each $n \in \mathbb{N}$, the constant functions $x \mapsto 2^n$ and $x \mapsto 0$ are measurable, hence by Exercise 1 g_n is measurable as well. Each $x \in [0,1]$ furnishes an integer N such that $x \geq 1/2^{N-1}$, whence $g_n(x) = 0$ for all $n > N$. It follows that $(g_n)_{n=1}^{\infty}$ converges pointwise to 0. Finally,

$$\int_{[0,1]} g_n(x)\, dx = 2^n \cdot \left(\frac{1}{2^{n-1}} - \frac{1}{2^n} \right) = 1$$

for all $n \in \mathbb{N}$, which implies that

$$\lim \int_{[0,1]} g_n(x)\, dx = 1 \neq 0 = \int_{[0,1]} \lim_{n\to\infty} g_n(x)\, dx,$$

as was to be shown. □