SUGGESTED SOLUTIONS FOR PROBLEM SET 10

SPRING 2011, MATH 312:01

Exercise 1. Let f be a uniformly continuous function defined on $\mathbb{R}$, and, for each natural number n, let $f_n(x) = f(x + 1/n)$. Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.

Proof. We fix $\varepsilon > 0$, and find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for every $x, y \in \mathbb{R}$. Then, for any $n > 1/\delta$ and $x \in \mathbb{R}$, we have

$$|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| < \varepsilon,$$

whence $f_n \to f$ uniformly. □

Exercise 2. Let f be the continuous function defined on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0; \\ 1 & \text{for } x = 0. \end{cases}$$

Show that f is infinitely differentiable for all x by showing that the Taylor series for $\sin x$, when divided term-by-term by x, results in a new power series with radius of convergence ∞, which must therefore be the Taylor series for f.

Proof. Recall that the Taylor series of the sine function is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

so that

$$f(x) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}.$$

Note that plugging in $x = 0$ in the above series gives $f(0) = 1$, which agrees with the definition of $f(x)$. Since

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)x^2}{(2n+3)(2n+2)} \right| = 0 < 1$$

for any $x \in \mathbb{R}$, it follows from the ratio test that the above series converges (uniformly) everywhere. The series can thus be differentiated term-by-term, which then results in another differentiable power series, and we conclude that f is infinitely differentiable. □

Remark (Differentiability classes). A continuous function is also known as a function of ***class*** C^0. If a function has a continuous derivative, then the function is of ***class*** C^1. In general, if a function is r-times differentiable with a continuous rth derivative, then the function is of ***class*** C^r. If a function has derivatives of all orders, then the function is of ***class*** C^∞, also referred to as a ***smooth function*** or a C^∞***-function***. If a function has a Taylor-series expansion, then the function is of ***class*** C^ω: such a function is called an ***analytic function***.

For real-valued functions on the real line, there is a strict-inclusion relationship between different differentiabilty classes. Not every continuous function is differentiable. Not every differentiable function is infinitely differentiable. Not every smooth function is analytic. Indeed,

$$C^0 \supsetneq C^1 \supsetneq \cdots \supsetneq C^r \supsetneq \cdots \supsetneq C^\infty \supsetneq C^\omega.$$

Exercise 3. Recall that Fourier coefficients for a bounded measurable function defined on $[-\pi, \pi]$ are given by

$$\begin{aligned} a_0 &= \int_{[-\pi,\pi]} \frac{f(x)}{\sqrt{2\pi}} \\ a_j &= \int_{[-\pi,\pi]} \frac{f(x)\cos(jx)}{\sqrt{\pi}} \\ b_j &= \int_{[-\pi,\pi]} \frac{f(x)\sin(jx)}{\sqrt{\pi}} \end{aligned}$$

(1) Show that if f is an even function, then the coefficients b_j are 0 and the coefficients a_j are given by $2\int_{[0,\pi]} f(x)\cos(jx)/\sqrt{\pi}$.
(2) Show that if f is an odd function, then the coefficients a_j are 0 and the coefficients b_j are given by $2\int_{[0,\pi]} f(x)\sin(jx)/\sqrt{\pi}$.

Proof. We first show that the product of two even functions is even, the product of an even function and an odd function is odd, and the product of two odd functions is even. To this end, we suppose that $g_1(x)$ and $g_2(x)$ are even, and $g_3(x)$ and $g_4(x)$ odd. Then the product of two even functions is even:

$$(g_1g_2)(-x) = g_1(-x)g_2(-x) = g_1(x)g_2(x) = (g_1g_2)(x)$$

the product of an even function and an odd function is odd:

$$(g_2g_3)(-x) = g_2(-x)g_3(-x) = g_2(x)(-g_3(x)) = -(g_2g_3)(x)$$

and the product of two odd functions is odd:

$$(g_3g_4)(-x) = g_3(-x)g_4(-x) = (-g_3(x))(-g_4(x)) = (g_3g_4)(x).$$

Recall that $\cos(jx)/\sqrt{\pi}$ is even, and $\sin(jx)/\sqrt{\pi}$ odd. We also recall Exercise 2, Problem set 4:

Theorem. *Suppose f is integrable on $[-b, b]$.*

$$\int_{-b}^{b} f\,dx = 0$$

if f is odd, and

$$\int_{-b}^{b} f\,dx = 2\int_{0}^{b} f\,dx$$

if f is even.

If f is even, then $f(x)\cos(jx)/\sqrt{\pi}$ is even, and so

$$a_j = \int_{[-\pi,\pi]} \frac{f(x)\cos(jx)}{\sqrt{\pi}} = 2\int_{[0,\pi]} \frac{f(x)\cos(jx)}{\sqrt{\pi}}.$$

Moreover, $f(x)\sin(jx)/\sqrt{\pi}$ is odd, and so

$$b_j = \int_{[-\pi,\pi]} \frac{f(x)\sin(jx)}{\sqrt{\pi}} = 0.$$

If f is odd, then $f(x)\cos(jx)/\sqrt{\pi}$ is odd, and so

$$a_j = \int_{[-\pi,\pi]} \frac{f(x)\cos(jx)}{\sqrt{\pi}} = 0.$$

Moreover, $f(x)\sin(jx)/\sqrt{\pi}$ is even, and so

$$b_j = \int_{[-\pi,\pi]} \frac{f(x)\sin(jx)}{\sqrt{\pi}} = 2\int_{[0,\pi]} \frac{f(x)\sin(jx)}{\sqrt{\pi}}.$$

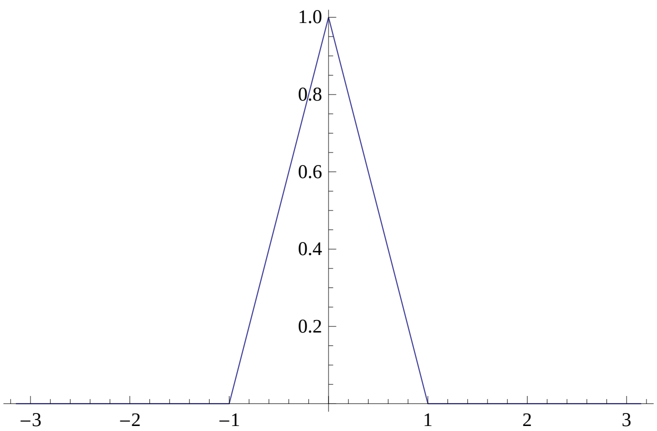
This completes the proof. □

Exercise 4. Let f be a continuous function defined on $[-\pi,\pi]$ as follows:

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 1 \\ 1+x & \text{for } -1 \le x \le 0 \\ 1-x & \text{for } 0 \le x \le 1 \\ 0 & \text{for } 1 \le x \le \pi. \end{cases}$$

Compute the Fourier coefficients for f.

Proof. Note that f is even; no, really:



By Exercise 3, $b_j = 0$ for all $j \in \mathbb{N}$. If $j = 0$, then

$$\begin{aligned} a_0 &= \int_{[-\pi,\pi]} \frac{f(x)}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{0} 1 + x\,dx + \int_0^1 1 - x\,dx\right) \\ &= \frac{1}{\sqrt{2\pi}}, \end{aligned}$$

for $\int_{-1}^{0} 1+x\,dx + \int_0^1 1-x\,dx$ is the area of the triangle. If $j \neq 0$, then

$$\begin{aligned}
a_j &= \int_{[-\pi,\pi]} \frac{f(x)\cos(jx)}{\sqrt{\pi}} \\
&= 2\int_{[0,\pi]} \frac{f(x)\cos(jx)}{\sqrt{\pi}} \\
&= \frac{2}{\sqrt{\pi}} \int_0^1 (1-x)\cos(jx)\,dx \\
&= \frac{2}{\sqrt{\pi}} \left(\left.\frac{(1-x)\sin(jx)}{j}\right|_0^1 + \int_0^1 \frac{\sin(jx)}{j}\,dx \right) \\
&= \frac{2}{\sqrt{\pi}} \cdot -\left.\frac{\cos(jx)}{j^2}\right|_0^1 \\
&= \frac{2}{\sqrt{\pi}} \cdot \frac{1-\cos j}{j^2} \\
&= \frac{2-2\cos j}{j^2\sqrt{\pi}}.
\end{aligned}$$

□