Challenge Problem Set 4, Math 292 Spring 2012

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1 Introduction

This challenge problem set is about using changes of variables based on symmetries to solve non-linear systems.

Let U be the open positive quadrant in $\mathbb{R}^2$:

$$U = \{(x,y)\ :\ x > 0\ ,\ y > 0\ \}\ .$$

Consider the vector field $\mathbf{v}(x,y)$ define on U given by

$$\mathbf{v}(x,y) = \left(\sqrt{\frac{x}{y}}, \sqrt{\frac{y}{x}}\right)\ .$$

Our goal is to find a change of variables that makes it very easy to solve for the general solution of the system of equations

$$(x(t),y(t))' = \mathbf{v}(x(t),y(t))\ . \tag{1.1}$$

1: *For each $u \in \mathbb{R}$, let g^u denote the matrix exponential e^{uA} where $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then*

$$g^u = \begin{bmatrix} e^{-u} & 0 \\ 0 & e^u \end{bmatrix}\ .$$

Show that for each u, the transformation sending $\mathbf{x}$ to $g^u\mathbf{x}$ is a one-to-one transformation from U onto U.

By the group properties of the matrix exponential; i.e., $e^{0A} = I$ and $e^{tA}e^{uA} = e^{(t+u)A}$, the family of transformations $\{g^u\}$ is a one parameter transformation group of U.

2: *Compute $g^u_*\mathbf{v}(x,y)$, the induced transformation of the vector field $\mathbf{v}$, recalling that*

$$g^u_*\mathbf{v}(x,y) = [D_{g^u}(g^{-u}(x,y))]\mathbf{v}(g^{-u}(x,y))\ .$$

That is, compute the Jacobian matrix of the transformation g^u, which is easy since the transformation is linear, multiply $\mathbf{v}$ by this Jacobian matrix , and evaluate this product is the point $g^{-u}(x,y)$.

Using your result, show that the transformation group $\{g^u\}$ is a symmetry group of the vector field $\mathbf{v}$.

Now that we have a symmetry group, we can construct a coordinate system from it that will render the system (1.1) easy to solve. The "usual recipe" gives the inverse coordinate transformation

$$h^{-1}(x,y) = g^u\mathbf{x}(v) \tag{1.2}$$

Where $\mathbf{x}(v)$ is some curve in U that intersects each orbit exactly once. To keep the cooridinates simple, we chose this curve a simply as possible. Given $(x_0, y_0) \in U$, the orbit of $\{g^u\}$ passing through (x_0, y_0) is, by definition, the curve parameterized by

$$\mathbf{x}(u) = g^u(x_0, y_0) = (e^{-u}x_0, e^u y_0) \ .$$

The points on this curve are exactly the points in U on the hyperbola

$$xy = x_0 y_0 \ .$$

3: *Sketch a few of these hyperbolas. Show that the curve parameterized by* $\mathbf{x}(v) = (v,v)$, $v \geq 0$, *which runs "up the diagonal" in U intersects each orbit exactly once. Then show that with this choice* $\mathbf{x}(v)$, *the change of coordinates h defined by (1.2) is given explicitly by*

$$x = e^{-u}v \qquad \text{and} \qquad y = e^u v \ , \tag{1.3}$$

so that

$$u(x,y) = \frac{1}{2}\ln\left(\frac{y}{x}\right) \qquad \text{and} \qquad v(x,y) = \sqrt{xy} \ . \tag{1.4}$$

Here, u ranges over all of $\mathbb{R}$, and v ranges over $(0,\infty)$, *so h transforms U onto V, the "right half plane" , given by*

$$V = \{\ (u,v)\ :\ v > 0\ \} \ .$$

The theory we have developed in class guarantees us that the equivalent transformed system

$$(u(t), v(t))' = (h_*\mathbf{v})(u(t), v(t)) \ . \tag{1.5}$$

will be easy to solve.

4: *Compute the transformed vector field* $(h_*\mathbf{v})(u(t), v(t))$, *using*

$$(h_*\mathbf{v})(u(t), v(t)) = [D_h(h^{-1}(u,v)]\mathbf{v}(h^{-1}(u,v)) \ ,$$

and then ind the general solution of (1.5). Using this result, find the general solution of (1.1).

5: *Show that a vector field* $\mathbf{w}(x,y) = (a(x,y), b(x,y)$ *on U satisfies*

$$g^u_*\mathbf{w} = \mathbf{w}$$

for all u if and only if

$$(e^u a(e^u x, e^{-u}y), e^{-u}b(e^u x, e^{-u}y)) = (a(x,y), b(x,y)$$

for all u, x and y, and that this is true if and only if

$$\begin{aligned} a(x,y) &= -x\frac{\partial}{\partial x}a(x,y) + x\frac{\partial}{\partial y}a(x,y) \\ b(x,y) &= x\frac{\partial}{\partial x}b(x,y) - x\frac{\partial}{\partial y}b(x,y)\ . \end{aligned}$$

This gives you a class of nonlinear systems that can be solved by the change of variables used here.