Challenge Problem Set 5, Math 292 Spring 2012

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1 Introduction

This challenge problem set is about solving the wave equation.

$$\frac{\partial^2}{\partial t^2}h(t,y) = \frac{\partial^2}{\partial y^2}h(t,y) \tag{1}$$

with wave speed $c = 1$ subject to the boundary conditions

$$h(t,0) = h(t,L) = 0\ , \tag{2}$$

suitable for modeling a clamped vibrating string with uniform linear mass density.

1: Show that the functions

$$h_0(y) = \sin^3(\pi y/L) \qquad \text{and} \qquad v_0(y) = 1 - \cos^4(\pi y/L)$$

may be expressed as trigonomentric polynomials of the form

$$\sum_{n=1}^{N} a_n \sin(n\pi y/L)$$

for some finite N and some coefficients a_n, $n = 1,\dots,N$. Find the N and the coefficients in each case.

2: Continung with the above problem, solve the wave equation (1) subject to the boundary conditions (2) for the initial data

$$h(0,y) = h_0(y) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(),y) = v_0(y)$$

for all $y\ in[0,L]$ with h_0 and v_0 as in problem 1. That is write down an explicit expression for the solution at time t.

Using any sort of graphing calculator, or Maple or whatever you find convenient, produce graphs of the solution at time $t = 0$, $t = L/2$ and $t = L$.

3: Now consider the function $h_0(y) = y(L-y)$. This cannot be expressed as a finite trigonmetric polynomial, but it does have a representation as a trigonometric series in the form

$$h_0(y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y/L)$$

Accepting that this is the case, use the "orthgonality relations" proved in class; i.e.,

$$\int_0^L \sin(n\pi y/L)\sin(m\pi y/L) = \begin{cases} L/2 & m = n \\ 0 & m \neq n \end{cases} .$$

find the "Fourrienr coefficients" a_n for this function. Show that

$$\lim_{n\to\infty} a_n = 0 .$$

For $L = 1$, find an N large enough that

$$\left| h_0(y) - \sum_{n=N}^{\infty} a_n \sin(n\pi y/L) \right| \leq 10^{-2} .$$

4: (Extra Credit) This is not hard, but is will require the use of Maple or Mathematica or some good programable calculator. Use the approximation fo $h_0(y) = y(L-y)$ found in the previous problem by a trigonmentric polynomial, and solve the wave equation for $L = 1$, this h_0 and $v_0 = 0$, and sxplcitly graph the solutions for $t = 0$, $t = 1/2$ and $t = 1$.