SUGGESTED SOLUTIONS TO HOMEWORK SET 5

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Problem 1. (a) Let U be an open subset of $[-\infty, \infty]$. As f is $\mathcal{M}$-measurable, the preimage $f^{-1}(U)$ is $\mathcal{M}$-measurable. Now, $(f|_E)^{-1}(U) = f^{-1}(U) \cap E$, and so $(f|_E)^{-1}(U)$ is $\mathcal{M}_E$-measurable.

Let $(s_n^+)_{n=1}^\infty$ be a sequence of nonnegative $\mathcal{M}$-simple functions that increase pointwise to f^+ (Definition 88.2). Repeating the argument presented in the above paragraph, we see that $(s_n|_E)_{n=1}^\infty$ is a sequence of nonnegative $\mathcal{M}_E$-simple functions that increase pointwise to $f^+|_E$. Since f^+ is measurable, another repetition of the argument above shows that $f^+|_E$ is measurable.

For each $n \in \mathbb{N}$, we write

$$s_n^+ = \sum_{m=1}^M a_m^n \chi_{E_m^n},$$

where $a_1^n, \ldots, a_m^n \in (0, \infty)$ and $E_1^n, \ldots, E_m^n \in \mathcal{M}$. Observe that

$$\int_E s_n^+|_E \, d\lambda = \sum_{m=1}^M \int_E \left(a_m^n \chi_{E_m^n}\right)|_E \, d\lambda = \sum_{m=1}^M \int_E a_m^n \chi_{E_m^n \cap E} \, d\lambda = \sum_{m=1}^M a_m^n \lambda(E_m^n \cap E)$$

by the definition of characteristic functions. Since $\lambda(A) = \mu(A \cap E)$ for all $A \in \mathcal{M}$, we see that

$$\lambda(E_m^n \cap E) = \mu\left((E_m^n \cap E) \cap E\right) = \mu(E_m^n \cap E).$$

Since $\chi_{E_m^n \cap E} = \chi_{E_m^n}\chi_E$, it follows that

$$\begin{aligned}\int_E s_n^+|_E \, d\lambda &= \sum_{m=1}^M a_m^n \mu(E_m^n \cap E) = \sum_{m=1}^M a_m^n \int_X \chi_{E_m^n}\chi_E \, d\mu \\ &= \int_X \left(\sum_{m=1}^M a_m^n \chi_{E_m^n}\right)\chi_E \, d\mu = \int_X s_n^+ \chi_E \, d\mu.\end{aligned}$$

Our computations were independent of n, and so we invoke the monotone convergence theorem to conclude that

$$\int_E f^+|_E \, d\lambda = \lim_{n\to\infty} \int_E s_n^+|_E \, d\lambda = \lim_{n\to\infty} \int_X s_n^+ \chi_E \, d\mu = \int_X f^+ \chi_E \, d\mu.$$

An analogous argument shows that

$$\int_E f^-|_E \, d\lambda = \int_X f^- \chi_E \, d\mu.$$

Since $f \in \mathscr{L}(X, \mathcal{M}, \mu)$, it now follows that $f|_E \in \mathscr{L}(E, \mathcal{M}_E, \lambda)$. □

(b) Let U be an open subset of $[-\infty, \infty]$. As f is $\mathcal{M}_E$-measurable, the preimage $f^{-1}(U)$ is $\mathcal{M}_E$-measurable. By the definition of $\mathcal{M}_E$, we can find a $\mathcal{M}$-measurable set F such that $f^{-1}(U) = F \cap E$. Since E is $\mathcal{M}$-measurable, it follows that $f^{-1}(U)$

is $\mathcal{M}$-measurable. Now, $g^{-1}(U) = f^{-1}(U)$, and so $g^{-}(U)$ is $\mathcal{M}$-measurable. We conclude that g is $\mathcal{M}$-measurable.

Let us now observe that $g = f|_E$. The desired result now follows from repeating the argument we have presented above verbatim. □

Remark 1.1. The key technical points of this problem are:

(1) checking that restriction and extension preserve measurability of a function, and
(2) reconciling the fact that the restriction and the original function (or the extension and the original function) have different domains.

There is no content to this problem beyond these two, so considering obvious things obvious will, of course, result in a low grade.

Problem 2. 88.6. Since $\mu(X) < \infty$, the simple function $M\chi_X$ is integrable with $\int M\chi_X = m\mu(X)$. In particular, $|f_n| \leq M\chi_X$ for all $n \in \mathbb{N}$, and so the desired result now follows from the dominated convergence theorem. □

88.10. For each $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$. Then

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \leq \int_X |f_n - f| \, d\mu < \int_X \varepsilon \, d\mu = \varepsilon\mu(X).$$

The last quantity is finite, as $\mu(X) < \infty$. Since $\varepsilon > 0$ was arbitrary, the desired result now follows. □

Remark 2.1. It is interesting to note that every pointwise almost-everywhere convergent sequence of measurable functions on a finite-measure set is *nearly* uniformly convergent: this is *Egoroff's theorem*. This is to say that uniform convergence is not that much stronger than pointwise convergence in the context of Lebesgue theory on finite-measure sets. Indeed, we already have the dominated convergence theorem, which is more than enough for the most part.

Problem 3. Omitted. (Our mantra is: *integration does not see what happens on sets of measure zero.*)

Problem 4. Fix $f \in \mathscr{L}(X, \mathcal{M}, \mu)$. Given $n \in \mathbb{N}$, we let

$$E_n = \{x \in X : |f(x)| > n\} \text{ and } F_n = \{x \in X : |f(x)| \leq n\}.$$

Then, for each $A \in \mathcal{M}$ and every $n \in \mathbb{N}$ we have a *layer-cake decomposition*

$$|f|\chi_A = (|f|\chi_A)\chi_{E_n} + (|f|\chi_A)\chi_{F_n} = |f|\chi_{A \cap E_n} + |f|\chi_{A \cap F_n},$$

so that

$$\int_A |f| \, d\mu = \int_{A \cap E_n} |f| \, d\mu + \int_{A \cap F_n} |f| \, d\mu.$$

We shall estimate each integral separately, which will then give us an idea as to which A we should choose.

To estimate the first integral, we observe that $|f|$ is finite almost everywhere, for otherwise

$$\int_X |f| \, d\mu \geq \int_{\{x \in X : |f(x)| = \infty\}} |f| \, d\mu = \infty.$$

Since $|f|$ is finite almost everywhere, $|f|\chi_{E_n} \to 0$ almost everywhere, and so

$$\lim_{n \to \infty} \int_{E_n} |f| \, d\mu = 0.$$

We now observe that $|f|\chi_{E_n} \leq |f|$ for all $n \in \mathbb{N}$, whence it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_{E_n} |f|\, d\mu = \int_X \lim_{n\to\infty} |f|\chi_{E_n}\, d\mu = 0.$$

Therefore, if we fix $\varepsilon > 0$, then we can find an $n_0 \in \mathbb{N}$ such that

$$\int_{A\cap E_{n_0}} |f|\, d\mu \leq \int_{E_{n_0}} |f|\, d\mu < \frac{\varepsilon}{2}.$$

By noting that $|f| \leq n_0$ on F_{n_0}, we obtain the following estimate on the second integral:

$$\int_{A\cap F_{n_0}} |F|\, d\mu \leq \int_{A\cap F_{n_0}} n_0\, d\mu \leq n_0\, \mu(A \cap F_{n_0}).$$

Combining the two estimates, we obtain the upper bound

$$\int_A |f|\, d\mu < \frac{\varepsilon}{2} + n_0\, \mu(A \cap F_{n_0}),$$

independent of the choice of $A \in \mathcal{M}$. We can therefore pick $A \in \mathcal{M}$ with $\mu(A) < \frac{\varepsilon}{2n_0}$ to get the estimate

$$\int_A |f|\, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

Remark 4.1. The term *layer-cake decomposition* comes from the fact that we split up a function into two "layers". While the above proof deviates significantly from the hint, I included it here as the layer-cake decomposition is an extremely versatile technique that is well worth learning. The proper context for decomposition techniques such as this is *harmonic analysis*.

Problem 5. Set $g_n = f_1 - f_n$, so that $g_n \in \mathscr{L}(X, \mathcal{M}, \mu)$ and that $0 \leq g_n \leq g_{n+1}$ for all $n \in \mathbb{N}$. By the monotone convergence theorem,

$$\int_X f_1\, d\mu - \lim_{n\to\infty} \int_X f_n\, d\mu = \lim_{n\to\infty} \int_X g_n\, d\mu = \int_X f_1\, d\mu - \int_X f\, d\mu.$$

The desired result now follows.

As for the counterexample, we consider $f_n = \chi_{[n,\infty)}$ on $\mathbb{R}$ with the Lebesgue measure. Then $\int f_n\, d\mu = \infty$ for all $n \in \mathbb{N}$; in particular, f_1 is not integrable. Now, $f_n \to 0$ everywhere, and so

$$\infty = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)\, dx = 0.$$

It follows that $(f_n)_{n=1}^{\infty}$ is a desired counterexample. □

Problem 6. For each $x \in X$, the series $\sum f_n(x)$ converges absolutely by the hypothesis, and so it converges. It follows that $\sum f_n$ converges pointwise. Observe also that $|\sum_{n=1}^{N} f_n| \leq \sum |f_n|$ for all $N \in \mathbb{N}$, whence the dominated convergence theorem implies that

$$\int_X \sum_{n=1}^{\infty} f_n\, d\mu = \lim_{N\to\infty} \lim_X \sum_{n=1}^{N} f_n\, d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu$$

by the linearity of the Lebesgue integral. □

Problem 7. Fix $t > 0$. We prove Chebyshev's inequality as follows:

$$\begin{aligned}\mu(\{x : |f(x)| \geq t\}) &= \int_{\{x:|f(x)|\geq t\}} 1 \, d\mu = \int_{\{x:|f(x)|\geq t\}} \frac{t}{t} \, d\mu \\ &\leq \int_{\{x:|f(x)|\geq t\}} \frac{|f(x)|}{t} \, d\mu = \frac{1}{t} \int_{x:|f(x)|\geq t\}} |f(x)| \, d\mu \\ &\leq \frac{1}{t} \int_X |f(x)| \, d\mu.\end{aligned}$$

If $\int |f_n| \, d\mu \to 0$ as $n \to \infty$, then Chebyshev's inequality shows that

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - 0| \geq t\}) = 0$$

for all $t > 0$: in this case, we say that $(f_n)_{n=1}^{\infty}$ *converges to zero in measure*. It now follows from Exercise 87.13 that there exists a subsequence of $(f_n)_{n=1}^{\infty}$ that converges pointwise μ-almost everywhere to 0.

As for the required counterexample, we consider the sequence

$$\chi_{[0,1]}, \chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/3]}, \chi_{[1/3,2/3]}, \chi_{[2/3,1]}, \chi_{[0,1/4]}, \cdots.$$

It follows at once (why?) that this sequence converges to zero in measure. Nevertheless, given each point $x \in [0, 1]$, the sequence takes the values of 1 infinitely often at x, hence it cannot converge pointwise to 0. $\square$