CALCULUS II, SUMMER 2015 - ABSOLUTE CONVERGENCE

The two foundational convergence tests for series are applicable only to series with nonnegative terms:

Theorem 1 (Comparison test)**.** *Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences with nonnegative terms, and suppose that*

$$a_n \leq b_n$$

for all $n \geq 1$.
(1) If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty$ converges;
(2) If $\sum_{n=1}^\infty a_n$ diverges, then $\sum_{n=1}^\infty$ converges.

Theorem 2 (Integral test)**.** *Let $f : [1, \infty) \to \mathbb{R}$ be a continuous, nonnegative, decreasing function, and let $a_n = f(n)$ for each $n \in \mathbb{N}$.*
(1) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges;
(2) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

In dealing with series, it is therefore paramount that we understand the "nonnegative version" of it.

Definition 3. A series $\sum_{n=1}^\infty a_n$ is said to *converge absolutely* if

$$\sum_{n=1}^\infty |a_n|$$

converges. If $\sum_{n=1}^\infty a_n$ converges but $\sum_{n=1}^\infty |a_n|$ diverges, then we say that the series $\sum_{n=1}^\infty |a_n|$ *converges conditionally.*

In order to determine convergence of $\sum a_n$, it is sufficient to test the convergence of its nonnegative version $\sum |a_n|$.

Theorem 4. *If $\sum_{n=1}^\infty a_n$ converges absolutely, then $\sum_{n=1}^\infty a_n$ converges.*

Proof. We first observe that

$$0 \leq |a_n| - a_n \leq 2|a_n|$$

for all $n \in \mathbb{N}$. To see this, we note that $|a_n|$ is either a_n or $-a_n$ depending on the sign of a_n. If $|a_n| = a_n$, then

$$|a_n| - a_n = 0,$$

and, if $|a_n| = -a_n$, then

$$|a_n| - a_n = -2a_n = 2|a_n|.$$

Observe now that

$$\sum_{n=1}^\infty |a_n| - a_n$$

is a series with nonnegative terms, whose terms are bounded above by the terms of $\sum_{n=1}^\infty |a_n|$. Since $\sum_{n=1}^\infty |a_n|$ converges, it follows from the comparison test (Theorem 1) that $\sum_{n=1}^\infty |a_n| - a_n$ converges.

Since $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |a_n| - a_n$ are both convergent sequences, we see that

$$\sum_{n=1}^{\infty} |a_n| - \left(\sum_{n=1}^{\infty} |a_n| - a_n \right)$$

is a convergent sequence. We conclude from the algebraic limit theorem that

$$\sum_{n=1}^{\infty} |a_n| - \left(\sum_{n=1}^{\infty} |a_n| - a_n \right) = \sum_{n=1}^{\infty} a_n,$$

whence $\sum_{n=1}^{\infty} a_n$ converges. □

We note the contrapositive:

Corollary 5. *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.*

Absolute convergence is, however, strictly stronger than convergence: in other words, the notion of conditional convergence is nontrivial. The next example illustrates this point.

Example 6. The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test (or the Dirichlet test), but its nonnegative version $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges. It follows that the alternating harmonic series is conditionally convergent. □

The nonnegative version of an absolutely convergent series provides an estimate on the size of the series.

Theorem 7 (Infinite triangle inequality)**.** *For all series $\sum_{n=1}^{\infty} a_n$,*

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

The infinite triangle inequality provides further intuition as to why absolute convergence is a stronger property than convergence. Since the size of $\sum_{n=1}^{\infty} a_n$ is always bounded above by the size of $\sum_{n=1}^{\infty} |a_n|$, the finiteness of $\sum_{n=1}^{\infty} |a_n|$ must force the finiteness of $\sum_{n=1}^{\infty} a_n$. (A rigorous version of this argument makes use of the *Cauchy criterion for convergence*, which is beyond the scope of this course,)

There are two standard tests for absolute convergence.

Theorem 8 (Ratio test)**.** *Consider a series $\sum_{n=1}^{\infty} a_n$.*

(1) If

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(2) If

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1,$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

(3) In all other cases, the test is inconclusive: the series could converge conditionally, converge absolutely, or diverge.

Theorem 9 (Root test)**.** *Consider a series $\sum_{n=1}^{\infty} a_n$.*

(1) *If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1,$$

then $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(2) *If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1,$$

then $\sum_{n=1}^{\infty} a_n$ *diverges.*

(3) *In all other cases, the test is inconclusive: the series could converge conditionally, converge absolutely, or diverge.*

See Section 8.4 in Stewart for examples.