**CALCULUS II, SUMMER 2015 - THE BINOMIAL THEOREM**

Recall that the *binomial theorem* yields the formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

for each positive integer $n$, where the coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are commonly known as the *binomial coefficients*. If we set $y = 1$, then we obtain the one-variable polynomial

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

We shall generalize the binomial theorem to establish a series representation of

$$(1+x)^\alpha,$$

where $\alpha$ is an arbitrary real number.

To this end, we introduce the *Pochhammer symbol*

$$(\alpha)_k = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$$

for each real number $\alpha$ and positive integer $k$, so that the binomial coefficients for a fixed positive integer $n$ can be written as follows:

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

This, in turn, implies that

$$(1+x)^n = \sum_{k=0}^{n} \frac{(n)_k}{k!} x^k.$$

In fact, if $k > n$, then

$$(n)_k = n(n-1)\cdots(n-n)(n-n-1)\cdots(n-k+1),$$

and so $(n)_k = 0$. Therefore, we have the following infinite-series representation

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} x^k,$$

provided that $n$ is a positive integer.

Taking a cue from this, we consider the *binomial series*

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k,$$

*Date*: August 10, 2015.

where $\alpha$ is an arbitrary real number. By setting $a_k = \frac{(\alpha)_k}{k!}$ for each $k \geq 0$, we see that the binomial series is of the form

$$\sum_{k=0}^{\infty} a_n x^k,$$

i.e., a power series centered at zero. Observe that

$$\frac{a_{k+1}}{a_k} = \frac{(\alpha)_{k+1}/(k+1)!}{(\alpha)_k/k!} = \frac{\alpha - k}{k+1},$$

and so the radius of convergence is

$$R = \left( \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \right)^{-1} = 1.$$

Let us, then, claim the best we could possibly hope for: we claim that we have the identity

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k$$

for each $\alpha \in \mathbb{R}$ and every $x \in (-1, 1)$. To prove this, we shall show that both $f(x) = (1+x)^\alpha$ and $g(x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k$ are solutions to the first-order linear ODE

$$y' - \frac{\alpha}{1+x} y = 0 \tag{1}$$

with initial condition $y(0) = 1$ on the interval $(-1, 1)$. It then follows from the uniqueness of solutions of first-order linear ODEs that $f = g$, as desired.

We first observe that

$$f'(x) = \alpha(1+x)^{\alpha-1} = \frac{\alpha}{1+x}(1+x)^\alpha = \frac{\alpha}{1+x} f(x)$$

for all $x \neq 1$. Moreover,

$$f(0) = (1+0)^\alpha = 1,$$

and so $f$ is a solution to (1) for all $x \in \mathbb{R}$.

Now,

$$g'(x) = \sum_{k=1}^{\infty} \frac{k(\alpha)_k}{k!} x^{k-1} = \sum_{k=0}^{\infty} \frac{(k+1)(\alpha)_{k+1}}{(k+1)!} x^k = \sum_{k=0}^{\infty} \frac{(\alpha)_{k+1}}{k!} x^k$$

for all $-1 < x < 1$. Observe that

$$\frac{(\alpha)_{k+1}}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!} = \frac{(\alpha)_k(\alpha-k)}{k!},$$

and so

$$g'(x) = \sum_{k=0}^{\infty} \frac{(\alpha-k)(\alpha)_k}{k!} x^k$$

for all $-1 < x < 1$.

Since

$$g'(x) = \sum_{k=0}^{\infty} \frac{(k+1)(\alpha)_{k+1}}{(k+1)!} x^k = \sum_{k=0}^{\infty} \frac{(\alpha-k)(\alpha)_k}{k!} x^k,$$

we see that

$$(1+x)g'(x) = \sum_{k=0}^{\infty}(\alpha-k)\frac{(\alpha)_k}{k!}x^k + \sum_{k=0}^{\infty}(k+1)\frac{(\alpha)_{k+1}}{(k+1)!}x^{k+1}$$
$$= \sum_{k=0}^{\infty}(\alpha-k)\frac{(\alpha)_k}{k!}x^k + \sum_{k=1}^{\infty}k\frac{(\alpha)_k}{k!}x^k.$$

If $k=0$, then

$$\sum_{k=1}^{\infty}k\frac{(\alpha)_k}{k!}x^k = 0,$$

and so

$$\sum_{k=1}^{\infty}k\frac{(\alpha)_k}{k!}x^k = \sum_{k=0}^{\infty}k\frac{(\alpha)_k}{k!}x^k.$$

It now follows that

$$(1+x)g'(x) = \sum_{k=0}^{\infty}(\alpha-k)\frac{(\alpha)_k}{k!}x^k + \sum_{k=1}^{\infty}k\frac{(\alpha)_k}{k!}x^k$$
$$= \sum_{k=0}^{\infty}(\alpha-k)\frac{(\alpha)_k}{k!}x^k + \sum_{k=0}^{\infty}k\frac{(\alpha)_k}{k!}x^k$$
$$= \sum_{k=0}^{\infty}\alpha\frac{(\alpha)_k}{k!}x^k = \alpha g(x),$$

whence

$$g'(x) - \frac{\alpha}{1+x}g(x) = 0$$

for all $-1 < x < 1$.

Now,

$$g(0) = \frac{(\alpha)_0}{0!} = 1,$$

and so $g$ is a solution to (1) for all $-1 < x < 1$. We conclude from the uniqueness of solutions of first-order linear ODEs that

$$f(x) = g(x)$$

for all $-1 < x < 1$. In other words, the power-series expansion

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}x^k$$

is valid on the interval $(-1, 1)$.