CALCULUS II, SUMMER 2015 - DIAGNOSTIC TEST

This worksheet outlines the prerequisite material from precalculus and calculus 1. You are expected to be familiar with every topic covered here by the first lecture, on July 6, so please make an effort to go through the worksheet in advance. You do not have to write out the solutions, nor do you have to submit your answers to the diagnostic test. Please ask yourself the question

`Could I do this problem if I had to?`

for each problem, and give yourself an honest answer; review the relevant sections whenever your answer is a `no`. The parentheses next to problem number indicates the sections in the course textbook that covers the material.

1. Precalculus

Please go through the `diagnostic tests` section in pages xvii - xxii of the textbook.

2. Sets and Functions

Problem 2.1 (Set theory notes)**.** What is a set?

Problem 2.2 (Set theory notes)**.** Write out the following sets in roster notation:
(1) $\{x : x < 5 \textit{ and } x \text{ is a natural number}\}$
(2) $\{1, 3, 7, 2, 5\} \cap \{9, 18, 25\}$
(3) $\{x : x > 7 \textit{ and } x \text{ is an integer}\} \cap \{x : x < 10 \textit{ and } x \text{ is an integer}\}$
(4) $\{5, 7, 9\} \cup \{0\}$
(5) $[2, 5] \cap \{x : x \text{ is an integer}\}$
(6) $(-2, 7) \cap \{x : x \text{ is a natural number}\}$

Problem 2.3 (Set theory notes)**.** Write out the following sets in interval notation:
(1) $\{x : x > 7 \textit{ and } x \leq 15\}$
(2) $\{x : x < 9\}$
(3) $[1, 5] \cap (2, 7)$
(4) $(2, \infty) \cup (-\infty, 7)$
(5) $(2, 7) \cup [7, 10)$
(6) $[1, 5) \cup [2, 8)$

Problem 2.4 (§1.1)**.** What is a function? What is the domain of a function? What is the range of a function?

Problem 2.5 (§1.1)**.** What is an even function? What is an odd function?

Problem 2.6 (§1.1)**.** What does it mean for a function to be increasing on an interval? What does it mean for a function to be decreasing on an interval?

Problem 2.7 (§3.2)**.** What does it mean for a function to be one-to-one? What is the inverse function of a one-to-one function?

Date: June 28, 2015.

Problem 2.8 (§1.1, §3.1, §3.2, §3.5)**.** State the domain and range of the following functions.

(1) Linear function $f(x) = mx + b$
(2) The exponential function $g(x) = e^x$
(3) The sine function $h(x) = \sin x$
(4) The cosine function $f(x) = \cos x$
(5) The tangent function $g(x) = \tan x$
(6) The secant function $h(x) = \sec x$
(7) The cosecant function $f(x) = \csc x$
(8) The cotangent function $g(x) = \cot x$

Compute the inverses of (1)-(8) and state their domains and ranges.

3. Limits and Continuity

Problem 3.1 (§1.3, §1.5)**.** What does it mean for a function f to be continuous at a number a? What does it mean for f to be continuous on an interval?

Problem 3.2 (§1.3, §1.4, §1.5, §3.1, §3.2, §3.5)**.** Determine whether the functions below are continuous at 5.

(1) $f(x) = x^4 + 3x^2 + 5x + 7$
(2) $g(x) = \frac{x-3}{x^2-4x+4}$
(3) $h(x) = \frac{x^3-7x+5}{x^2-6x+5}$
(4) $h(x) = e^{5x^2-2} + 4$
(5) $f(x) = \ln(x - 6)$
(6) $g(x) = \cos^{-1} \frac{\pi x}{10}$

Problem 3.3 (§1.5)**.** State the intermediate value theorem.

Problem 3.4 (§1.5)**.** Solve Exercise 1.5.51 in the course textbook, quoted below:

> A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of the day on both days.

Problem 3.5 (§1.6)**.** Explain the following notations:

(1) $\lim_{x\to a} f(x) = \infty$
(2) $\lim_{x\to a} f(x) = -\infty$
(3) $\lim_{x\to\infty} f(x) = L$
(4) $\lim_{x\to\infty} f(x) = \infty$
(5) $\lim_{x\to\infty} f(x) = -\infty$
(6) $\lim_{x\to-\infty} f(x) = L$
(7) $\lim_{x\to-\infty} f(x) = \infty$
(8) $\lim_{x\to-\infty} f(x) = -\infty$

Problem 3.6 (§1.4)**.** State the squeeze theorem.

Problem 3.7 (§1.3, §1.4, §1.6, §3.7)**.** Compute the following limits:

(1) $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ (Hint: use the inequality $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$ and the squeeze theorem.)
(2) $\lim_{a\to\infty} \frac{1}{a^2}\sin a$ (Hint: use the inequality $-\frac{1}{a^2} \le \frac{1}{a^2}\sin a \le \frac{1}{a^2}$ and the squeeze theorem.)
(3) $\lim_{t\to 1} \frac{\sin \pi t}{2t}$
(4) $\lim_{y\to\frac{1}{2}} \tan \pi y$
(5) $\lim_{y\to\frac{1}{2}} |\tan \pi y|$
(6) $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$
(7) $\lim_{x\to\pi/2}(\sec x - \tan x)$

4. Differentiation

Problem 4.1 (§2.2)**.** What does it mean for a function f to be differentiable at a fixed number a?

Problem 4.2 (§2.2, §2.3, §2.4, §2.5)**.** Compute the derivatives of the following functions at 3.
(1) $f(x) = \sin^5 x \cos 5x$
(2) $g(x) = 3|x| + 5$
(3) $h(x) = \cos\sqrt{\cos(\tan(\pi(x-3)))} + 15e^{x^3+2x+1}$
(4) $f(x) = \frac{x^3+2x+1}{x^4+2x^3+3x^2+4x+5}$

Problem 4.3 (§2.6)**.** Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(-3, -4)$ by finding $\frac{dy}{dx}$ with respect to the expression $x^2 + y^2 = 25$.

Problem 4.4 (§2.7)**.** Solve Exercise 2.7.42 in the course textbook, quoted below:

> The minute hand on a watch is 8mm long and the hour hand is 4mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Problem 4.5 (§3.2)**.** State the inverse function theorem.

Problem 4.6 (§3.2, §3.3, §3.5)**.** Compute the derivatives of the following functions at 1.
(1) $f(x) = \tan^{-1}(x^2)$
(2) $g(\theta) = \arcsin\left(\sqrt{\sin\theta}\right)$
(3) $h(t) = \ln\ln\ln\ln 2t$
(4) $f(y) = \ln(e^{-y} + ye^{-y})$

Problem 4.7 (§4.1)**.** State the extremum value theorem. State Ferma's theorem on local extremum. State the definition of a critical number of a function f.

Problem 4.8 (§4.1)**.** Find the absolute maximum and absolute minimum values of the following functions on the given intervals, respectively:
(1) $f(x) = -3x^2 + 2x + 5$ on $[2, 7]$
(2) $g(x) = \frac{2x}{x^2-x+1}$ on $[0, 3]$
(3) $h(x) = x - \ln x$ on $[\frac{1}{2}, 2]$

Problem 4.9 (§4.2)**.** State the mean value theorem.

Problem 4.10 (§4.3)**.** Solve Exercises 4.3.1-4.3.10, quoted below:

> (a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

1. $f(x) = 2x^3 + 3x^2 - 36x$
2. $f(x) = 4x^3 + 3x^2 - 6x + 1$
3. $f(x) = x^4 - 2x^2 + 3$
4. $f(x) = \frac{x}{x^2+1}$
5. $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$
6. $f(x) = \cos^2 x - 2\sin x,\ 0 \le x \le 2\pi$
7. $f(x) = e^{2x} + e^{-x}$
8. $f(x) = x^2 \ln x$
9. $f(x) = x^2 - x - \ln x$
10. $f(x) = x^4 e^{-x}$

Problem 4.11 (§4.4)**.** Sketch the following curves:

(a) $y = \frac{4x}{\sqrt{x^2+1}}$
(b) $y = 2x\sqrt{3 - x^2} + 5$
(c) $y = 2x - 4\tan x,\ -\pi/2 < x < \pi/2$
(d) $y = \frac{3\sin x}{1+\cos x}$
(e) $y = x \ln x$

5. Integration

Problem 5.1 (§4.7, §5.2, §5.3)**.** Compute the following integrals:

(a) $\int_{-2}^{3} x^2 - 3\,dx$
(b) $\int_{\pi/4}^{\pi/3} 2\theta + 4\csc^2\theta\,d\theta$
(c) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{16}{1+x^2}\,dx$
(d) $\int_0^5 |4x - 2|\,dx$
(e) $\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt$
(f) $\int_0^{\pi} (10e^y + 5\cos y)\,dy$

Problem 5.2 (§5.4)**.** State the fundamental theorem of calculus.

Problem 5.3 (§5.4)**.** State the mean value theorem for integrals.

Problem 5.4 (§5.4)**.** Find the derivatives of the following functions:

(a) $f(x) = \int_2^{1/x} 3\arctan t\,dt$
(b) $g(x) = \int_{\sin x}^{\cos x} (1 + t^2)^{10}\,dt$
(c) $h(x) = \int_4^x e^{4y^2-y}\,dy$

Problem 5.5 (§5.4)**.** Show that the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

satisfies the differential equation

$$y' = 2xy + \frac{2}{\sqrt{\pi}}.$$

6. Solutions to Selected Problems

Problem 2.2.

(1) $\{1, 2, 3, 4\}$
(2) $\varnothing$
(3) $\{8, 9\}$
(4) $\{0, 5, 7, 9\}$
(5) $\{2, 3, 4, 5\}$
(6) $\{1, 2, 3, 4, 5, 6\}$

Problem 2.3.

(1) $(7, 15]$
(2) $(-\infty, 9)$
(3) $(2, 5]$
(4) $(-\infty, \infty)$
(5) $(2, 10)$
(6) $[1, 8)$

Problem 3.2.

(1) $f(5) = 732$ and
$$\lim_{x\to 5} f(x) = 732,$$
and so f is continuous at 5.
(2) $g(5) = \frac{2}{9}$ and
$$\lim_{x\to 5} g(x) = \frac{2}{9},$$
and so g is continuous at 5.
(3) h has a vertical asymptote at 5, and so h cannot be continuous at 5.
(4) $h(5) = e^{123} + 4$ and
$$\lim_{x\to 5} h(x) = e^{123} + 4,$$
and so h is continuous at 5.
(5) 5 is outside the domain of f, and so f cannot be continuous at 5.
(6) 5 is outside the domain of g, and so g cannot be continuous at 5.

Problem 3.4. Let t represent time in the interval from 7AM to 7PM; using the 24-hour clock, we could write the interval as $[7, 19]$. Let D represent the distance between the monastery and the top of the mountain. Let $f(t)$ represent the distance between the monk and the monastery at time t during the first trip, so that
$$\begin{cases} f(7) &= 0 \\ f(19) &= D. \end{cases}$$
Let $g(t)$ represent the distance between the monk and the monastery at time t during the second trip, so that
$$\begin{cases} g(7) &= D \\ g(19) &= 0. \end{cases}$$
Since f and g are both continuous functions, the difference $f - g$ is a continuous function. Now,
$$(f - g)(7) = f(7) - g(7) = 0 - D = -D < 0$$

and

$$(f-g)(19) = f(19) - g(19) = D - 0 = D > 0,$$

and so the intermediate value theorem implies that $f - g$ takes the value of 0 at, say, t_0. Since

$$0 = (f-g)(t_0) = f(t_0) - g(t_0),$$

we see that

$$f(t_0) = g(t_0).$$

This implies that, at time t_0 on both day one and day two, the monk will be at the same distance away from the monastery. In other words, At time t_0 on both day one and day two, the monk will be at the same spot on his usual path.

Problem 3.7.

(1) $x^2 \sin \frac{1}{x}$ fails to be continuous at 0, and so we cannot compute the limit by direct substitution. Observe that

$$\lim_{x\to 0} -x^2 = 0$$

and

$$\lim_{x\to 0} x^2 = 0.$$

Since

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

for all $x \in (-\infty, 0) \cup (0, \infty)$, it follows from the squeeze theorem that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

(2) Observe that

$$\lim_{a\to\infty} -\frac{1}{a^2} = 0$$

and

$$\lim_{a\to\infty} \frac{1}{a^2} = 0.$$

Since

$$-\frac{1}{a^2} \le \frac{1}{a^2}\sin a \le \frac{1}{a^2}$$

for all $a \in (-\infty, 0) \cup (0, \infty)$, it follows from the squeeze theorem that

$$\lim_{a\to\infty} \frac{1}{a^2}\sin a = 0.$$

(3)

$$\lim_{t\to 1} \frac{\sin \pi t}{2t} = \frac{\sin(\pi \cdot 1)}{2 \cdot 1} = \frac{\sin \pi}{2} = 0.$$

(4) $\lim_{y^-\to\frac{1}{2}} \tan \pi y = \infty$ and $\lim_{y^+\to\frac{1}{2}} \tan \pi y = -\infty$, and so the limit does not exist at $\frac{1}{2}$.

(5) $\lim_{y^-\to\frac{1}{2}} |\tan \pi y| = \infty$ and $\lim_{y^+\to\frac{1}{2}} |\tan \pi y| = \infty$, and so

$$\lim_{y\to\frac{1}{2}} |\tan \pi y| = \infty.$$

(6) There are a few ways of solving this problem. One way is via rearrangement:

$$\begin{aligned}\frac{\cos\theta - 1}{\theta} &= \frac{(\cos\theta - 1)(\cos\theta + 1)}{\theta(\cos\theta + 1)} = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} \\ &= \frac{-(1-\cos^2\theta)}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} \\ &= \left(\frac{\sin\theta}{\theta}\right)\left(\frac{-\sin\theta}{\cos\theta + 1}\right).\end{aligned}$$

Since

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,$$

we see that

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta + 1}\right) = 1\cdot\frac{0}{2} = 0.$$

Another way is to note that $\cos 0 = 1$, and so

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta - \cos 0}{\theta}$$

is precisely the derivative of $\cos\theta$ at 0. Since the derivative of $\cos\theta$ is $-\sin\theta$, we see that

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = -\sin 0 = 0.$$

Yet another way is to use L'Hôpital's theorem. Since $\lim_{\theta\to 0}\cos\theta - 1 = 0$ and $\lim_{\theta\to 0}\theta = 0$, we can use L'Hôpital's theorem to conclude that

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\frac{-\sin\theta}{1} = -\sin 0 = 0.$$

(7) Observe that

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1-\sin x}{\cos x}$$

Since

$$\begin{aligned}\frac{1-\sin x}{\cos x} &= \frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)} = \frac{1-\sin^2 x}{\cos x(1+\sin x)} \\ &= \frac{\cos^2 x}{\cos x(1+\sin x)} = \frac{\cos x}{\cos x}\cdot\frac{\cos x}{1+\sin x} = \frac{\cos x}{1+\sin x},\end{aligned}$$

it follows that

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{1+\sin x} = \frac{0}{1+1} = 0.$$

Of course, $\lim_{x\to\frac{\pi}{2}} 1-\sin x = 0$ and $\lim_{x\to\frac{\pi}{2}}\cos x = 0$, and so we could also use L'Hôpital's theorem:

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x} = 0.$$