CALCULUS II, SUMMER 2015 - EXAM 2 SOLUTIONS

`130 points total = 120 points + 10 extra credit points`

Problem 1 (20 points). Solve the following differential equation:

$$x^2y' + xy + 2y^2 = 0.$$

Solution. Note that the above equation is equivalent to

$$y' = \frac{-xy - 2y^2}{x^2}.$$

Set $f(x, y) = \frac{-xy-2y^2}{x^2}$ and observe that

$$f(tx, ty) = \frac{-(tx)(ty) - 2(ty)^2}{(tx)^2} = \frac{t^2(-xy - 2y^2)}{t^2x^2} = \frac{-xy - 2y^2}{x^2} = f(x, y)$$

for all choices of t, x, and y, and so the differential equation in question is a first-order homogeneous ODE.

We can therefore make the substitution $v = y/x$ and $y' = v'x + v$ to transform the above equation to the following:

$$v'x + v = \frac{-xy - 2y^2}{x^2} = \frac{x^2(-y/x - 2y^2/x^2)}{x^2} = -y/x - 2(y/x)^2 = -v - 2v^2.$$

The equation

$$v'x + v = -v - 2v^2$$

is separable:

$$\frac{1}{-2 - 2v^2}v' = \frac{1}{x}.$$

Integrating both sides, we obtain the following:

$$\frac{1}{2}\ln|v + 1| - \frac{1}{2}\ln|v| = \ln|x| + C.$$

This is equivalent to

$$\ln\sqrt{\left|\frac{v + 1}{v}\right|} = \ln|xe^C|,$$

whence

$$\frac{v + 1}{v} = (xe^C)^2.$$

We set $K = e^{2C}$ and rearrange the above relation to obtain

$$v = \frac{1}{Kx^2 - 1}.$$

Since $v = y/x$, we conclude that

$$y = \frac{x}{Kx^2 - 1}.$$

□

Date: July 30, 2015.

Problem 2 (20 points). An object of mass 1000kg is thrown downwards from a helicopter at altitude 1000m. If the initial velocity of the object is 10m/s, what is the altitude of the object after 10 seconds? How about the velocity? For simplicity's sake, assume that the gravitational constant g is 10, the air resistance constant k is 100, and that there is no significance force other than gravitation and air resistance acting on the object.

Hint: $e^{-1} \approx 0.4$.

Solution. Recall that the free-fall equation is of the form

$$mv' = mg - kv,$$

which is equivalent to

$$v' + \frac{k}{m}v = g,$$

Since $k = 100$, $m = 1000$, and $g = 10$, we can write the above equation as follows:

$$v' + \frac{1}{10}v = 10.$$

This is a nonhomogeneous first-order ODE, and so the solution is of the form

$$v = be^{-A(t)} + e^{-A(t)} \int_a^t e^{A(u)} Q(u)\, du,$$

where $a = 0$, $b = 10$, $P(t) = \frac{1}{10}$, $Q(t) = 10$, and $A(u) = \int_a^u P(x)\, dx$. Therefore,

$$\begin{aligned} v &= 10e^{-\frac{1}{10}t} + e^{-\frac{1}{10}t} \int_0^t 10e^{\frac{1}{10}t}\, du \\ &= 10e^{-\frac{1}{10}} + e^{-\frac{1}{10}t}\left(100e^{\frac{1}{10}t} - 100\right) \\ &= -90e^{-\frac{1}{10}t} + 100. \end{aligned}$$

The displacement function is computed by integrating the velocity function:

$$s = \int v(t)\, dt + C = 900e^{-\frac{1}{10}t} + 100t + C$$

Since $s(0) = 0$, we see that $C = -900$, and so

$$s = 900e^{-\frac{1}{10}t} + 100t - 900.$$

Now,

$$s(10) = 900e^{-1} + 1000 - 900 \approx 900 \cdot 0.4 + 100 = 460,$$

and so the altitude after 10 seconds is approximately

$$1000 - 460 = 540\text{m}.$$

The velocity after 10 seconds is

$$v(10) = -90e^{-1} + 100 \approx -90 \cdot 0.4 + 100 = 64\text{m/s}.$$

□

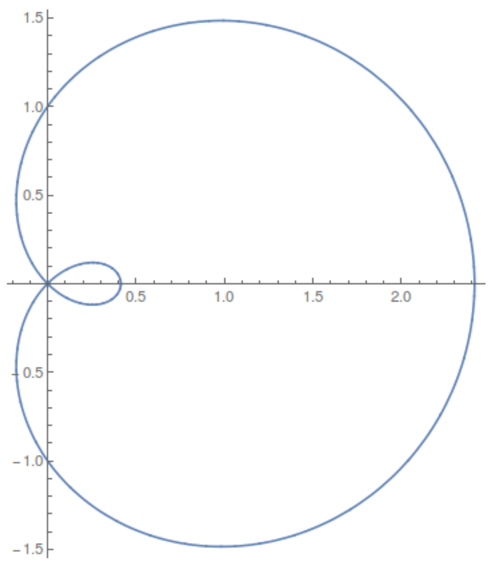
Problem 3 (20 points). Graph the following:

(1) $r = 1 + \sqrt{2}\cos\theta$, where $0 \le \theta \le \pi/2$;
(2) $r = 1 + \sqrt{2}\cos\theta$, where $0 \le \theta \le \pi$;
(3) $r = 1 + \sqrt{2}\cos\theta$, where $0 \le \theta \le 3\pi/2$;
(4) $r = 1 + \sqrt{2}\cos\theta$, where $0 \le \theta \le 2\pi$.

The resulting shape is known as a *limaçon.*

Hint: $\sqrt{2} \approx 1.4$.

Solution. Pay particular attention to the interval $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$.



□

Problem 4 (20 points). What is the area under the parametric curve

$$t \mapsto (-\tan t, \sec t)$$

on the interval $-\frac{\pi}{3} \le t \le \frac{\pi}{3}$?

Solution. $-\tan t$ is strictly decreasing on the interval, and so the area formula is

$$A = \int_{\pi/3}^{-\pi/3} y(t)x'(t)\,dt = \int_{\pi/3}^{-\pi/3} -\sec^3 t\,dt = \int_{-\pi/3}^{\pi/3} -\sec^3 t\,dt.$$

Observe that

$$A = \int_{-\pi/3}^{\pi/3} \sec^3 t\,dt = \int_{-\pi/3}^{\pi/3} (\tan^2 t + 1)\sec t\,dt = \int_{-\pi/3}^{\pi/3} \tan^2 \sec t\,dt + \int_{-\pi/3}^{\pi/3} \sec t\,dt$$

$$= \int_{-\pi/3}^{\pi/3} \tan^2 \sec t\,dt + \ln\left|\tan\left(\frac{\pi}{3}\right) + \sec\left(\frac{\pi}{3}\right)\right| - \ln\left|\tan\left(-\frac{\pi}{3}\right) + \sec\left(-\frac{\pi}{3}\right)\right|$$

$$= \int_{-\pi/3}^{\pi/3} \tan^2 \sec t\,dt + \ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right| = \int_{-\pi/3}^{\pi/3} \tan^2 \sec t\,dt + \ln\left|2+\sqrt{3}\right|^2.$$

It thus suffices to integrate

$$\int_{-\pi/3}^{\pi/3} \tan^2 \sec t \, dt.$$

We integrate by parts with $u = \tan t$ and $dv = \tan t \sec t \, dt$. We then have

$$\int_{-\pi/3}^{\pi/3} \tan^2 \sec t \, dt = \sec t \tan t \Big|_{t=-\pi/3}^{t=\pi/3} - \int_{-\pi/3}^{\pi/3} \sec^3 t \, dt = 4\sqrt{3} - \int_{-\pi/3}^{\pi/3} \sec^3 t \, dt.$$

Since

$$\int_{-\pi/3}^{\pi/3} \sec^3 t \, dt = \int_{-\pi/3}^{\pi/3} \tan^2 \sec t \, dt + \ln \left|2 + \sqrt{3}\right|^2,$$

it now follows that

$$A = \int_{-\pi/3}^{\pi/3} \sec^3 t \, dt = \frac{4\sqrt{3} + \ln \left|2 + \sqrt{3}\right|^2}{2} = 2\sqrt{3} + \ln(2 + \sqrt{3}).$$

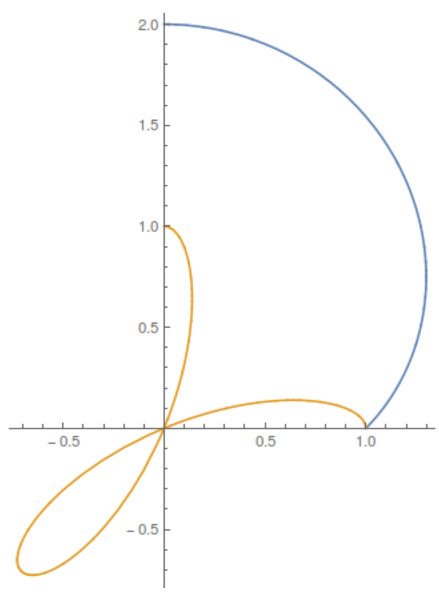
□

Problem 5 (20 points). Compute the area between the curves

$$r_1(\theta) = 1 + \sin\theta \quad \text{and} \quad r_2(\theta) = \cos 4\theta$$

on the interval $0 \le \theta \le \frac{\pi}{2}$.

Solution. Observe the following graph:



On the intervals $0 \le \theta \le \frac{\pi}{8}$ and $\frac{3\pi}{8} \le \theta \le \frac{\pi}{2}$, the regions are bounded above by r_1 and below by r_2. On the interval $\frac{\pi}{8} \le \theta \le \frac{3\pi}{8}$, the region is bounded by r_1 alone.

Therefore, the area of the region is

$$\begin{aligned}
A &= \frac{1}{2}\left(\int_0^{\pi/8} r_1^2 - r_2^2\,d\theta + \int_{\pi/8}^{3\pi/8} r_1^2\,d\theta + \int_{3\pi/8}^{\pi/2} r_1^2 - r_2^2\,d\theta\right) \\
&= \frac{1}{2}\left(\int_0^{\pi/2} r_1^2\,d\theta - \int_0^{\pi/8} r_2^2\,d\theta - \int_{3\pi/8}^{\pi/2} r_2^2\,d\theta\right) \\
&= \frac{1}{2}\left(\int_0^{\pi/2} (1+\sin\theta)^2\,d\theta - \int_0^{\pi/8} (\cos 4\theta)^2\,d\theta - \int_{3\pi/8}^{\pi/2} (\cos 4\theta)^2\,d\theta\right) \\
&= \frac{1}{2}\left(\int_0^{\pi/2} 1 + 2\sin\theta + \sin^2\theta\,d\theta - \frac{\pi}{8}\right) \\
&= \frac{1}{2}\left(\frac{\pi}{2} + 2 + \frac{\pi}{4} - \frac{\pi}{8}\right) = 1 + \frac{5\pi}{16}
\end{aligned}$$

□

Problem 6 (20 points). Find the length of one arch of the cycloid

$$x(t) = 3(t - \sin t);$$
$$y(t) = 3(1 - \cos t).$$

Solution.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^{2\pi} \sqrt{(3 - 3\cos t)^2 + (3\sin t)^2}\,dt \\
&= \int_0^{2\pi} \sqrt{9 - 18\cos t + 9\cos^2 t + 9\sin t^2}\,dt \\
&= 3\sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos t}\,dt \\
&= 3\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2(t/2)}\,dt \\
&= 6\int_0^{2\pi} \sin(t/2)\,dt \\
&= 24.
\end{aligned}$$

□

Bonus Problem (10 points). Prove the following uniqueness theorem:

Theorem. *Let P and Q be continuous functions on an open interval (α, β). Given two numbers a and b with $\alpha < a < \beta$, every solution to the differential equation*

$$y' + P(x)y = Q(x)$$

on the interval (α, β) with the initial condition $y(a) = b$ is given by the formula

$$y = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}\,dt,$$

where

$$A(x) = \int_a^x P(u)\, du.$$

Proof. Let $g(x)$ be a solution to the equation

$$y' + P(x)y = Q(x) \tag{1}$$

on the interval (α, β) with the initial condition $y(a) = b$. Set $h(x) = g(x)e^{A(x)}$ and observe that

$$\begin{aligned} h'(x) &= g'(x)e^{A(x)} + g(x)A'(x)e^{A(x)} \\ &= e^{A(x)}\left(g'(x) + A'(x)g(x)\right) \\ &= e^{A(x)}\left(g'(x) + P(x)g(x)\right). \end{aligned}$$

The last line follows from the fact that $A(x) = \int_a^x P(u)\, du$. g is a solution to (1), and so

$$h'(x) = e^{A(x)}Q(x).$$

The fundamental theorem of calculus implies that

$$h(x) - h(a) = \int_a^x e^{A(t)}Q(t)\, dt.$$

Let us compute

$$h(a) = g(a)e^{A(a)}.$$

Since g is a solution to (1) we have that $g(a) = b$. Moreover, $A(x) = \int_a^x P(u)\, du$, and so $A(a) = \int_a^a P(u)\, du = 0$. It now follows that

$$h(a) = b,$$

and so

$$h(x) = b + \int_a^x e^{A(t)}Q(t)\, dt.$$

Finally, $h(x) = g(x)e^{A(x)}$, and so

$$g(x) = be^{-A(x)} + e^{-A(x)}\int_a^x e^{A(t)}Q(t)\, dt,$$

as was to be shown. □