CALCULUS II, SUMMER 2015 - EXAM 2

130 points total = 120 points + 10 extra credit points

Name: ____________________________________ Score: ______/ 120

N.B. *Do not include scratch-work, but do write neatly and legibly. Everything written on these pages will be graded, so long as it is legible.*

Problem 1 (20 points). Solve the following differential equation:

$$x^2y' + xy + 2y^2 = 0.$$

Date: July 30, 2015.

Problem 2 (20 points). An object of mass 1000kg is thrown downwards from a helicopter at altitude 1000m. If the initial velocity of the object is 10m/s, what is the altitude of the object after 10 seconds? How about the velocity? For simplicity's sake, assume that the gravitational constant g is 10, the air resistance constant k is 100, and that there is no significance force other than gravitation and air resistance acting on the object.

Hint: $e^{-1} \approx 0.4$.

Problem 3 (20 points). Graph the following:

(1) $r = 1 + \sqrt{2}\cos\theta$, where $0 \leq \theta \leq \pi/2$;
(2) $r = 1 + \sqrt{2}\cos\theta$, where $0 \leq \theta \leq \pi$;
(3) $r = 1 + \sqrt{2}\cos\theta$, where $0 \leq \theta \leq 3\pi/2$;
(4) $r = 1 + \sqrt{2}\cos\theta$, where $0 \leq \theta \leq 2\pi$.

The resulting shape is known as a *limaçon*.

Hint: $\sqrt{2} \approx 1.4$.

Problem 4 (20 points). What is the area under the parametric curve

$$t \mapsto (-\tan t, \sec t)$$

on the interval $-\frac{\pi}{3} \leq t \leq \frac{\pi}{3}$?

Problem 5 (20 points). Compute the area between the curves

$$r_1(\theta) = 1 + \sin\theta \quad \text{and} \quad r_2(\theta) = \cos 4\theta$$

on the interval $0 \leq \theta \leq \frac{\pi}{2}$.

Problem 6 (20 points). Find the length of one arch of the cycloid

$$x(t) = 3(t - \sin t);$$
$$y(t) = 3(1 - \cos t).$$

Bonus Problem (10 points). Prove the following uniqueness theorem:

Theorem. *Let P and Q be continuous functions on an open interval (α, β). Given two numbers a and b with $\alpha < a < \beta$, every solution to the differential equation*

$$y' + P(x)y = Q(x)$$

on the interval (α, β) with the initial condition $y(a) = b$ is given by the formula

$$y = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}\, dt,$$

where

$$A(x) = \int_a^x P(u)\, du.$$

Differential Equations Reference Sheet

1. A *first-order linear ODE* is a differential equation of the form

$$y' + P(x)y = Q(x). \tag{1}$$

We say that (1) is *homogeneous* if $Q(x) = 0$ for all x and *nonhomogeneous* if $Q(x) \neq 0$ for some x. Given the initial condition

$$y(a) = b,$$

the solution of (1) is

$$y = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}\, dt,$$

where $A(x) = \int_a^x P(u)\, du$. In particular, if (1) is homogeneous, then the solution is given by the following simpler formula:

$$y = be^{-A(x)}.$$

2. A *first-order separable ODE* is a differential equation of the form

$$A(y)y' = Q(x). \tag{2}$$

Integrating both sides of (2) with respect to x, we obtain

$$\int A(y)\frac{dy}{dx}\, dx = \int Q(x)\, dx + C.$$

It then follows from the fundamental theorem of calculus that

$$\int A(y)\, dy = \int Q(x)\, dx + C.$$

Compute the antiderivatives and organize the terms to find the solution of (2).

3. A *first-order homogeneous ODE* is a differential equation of the form

$$y' = f(x, y), \tag{3}$$

where $f(tx, ty) = f(x, y)$ for all t, x, and y. The name comes from the fact that a two-variable function $f(x, y)$ satisfying the relation

$$f(tx, ty) = t^n f(x, y)$$

for all t, x, and y is said to be *homogeneous of degree n*. To solve (3), we make the substitution $v = y/x$, so that

$$y' = f(x, y) = f((1/x)x, (1/x)y) = f(1, y/x) = f(1, v).$$

Since $y' = v'x + vx' = v'x + v$, it follows that

$$v'x + v = f(1, v).$$

Consolidating the terms, we obtain the first-order separable ODE

$$\frac{1}{f(1, v) - v}v' = \frac{1}{x}.$$

4. A *homogeneous second-order ODE with constant coefficients* is a differential equation of the form

$$y'' + ay' + by = 0. \tag{4}$$

If $a = 0$, then the general solution of (4) takes the following form:

$$y = \begin{cases} C_1 e^{\sqrt{|b|}x} + C_2 e^{-\sqrt{|b|}x} & \text{if } b < 0; \\ C_1 + C_2 x & \text{if } b = 0; \\ C_1 \sin(\sqrt{|b|}x) + C_2 \cos(\sqrt{|b|}x) & \text{if } b > 0. \end{cases}$$

If $a \neq 0$, then we make the substitution

$$y = e^{-ax/2} u \tag{5}$$

to obtain

$$u'' + \frac{4b - a^2}{4} u = 0. \tag{6}$$

We solve (6) and use the relation (5) to obtain the solutions of (4). Carrying out this process, we see that the general solution of (4) takes the following form; for this purpose, we let $d = 4b - a^2$ and $k = \frac{1}{2}\sqrt{|d|}$:

$$y = \begin{cases} e^{-ax/2}\left(C_1 e^{kx} + C_2 e^{-kx}\right) & \text{if } d < 0; \\ e^{-ax/2}\left(C_1 + C_2 x\right) & \text{if } d = 0; \\ e^{-ax/2}\left(C_1 \sin kx + C_2 \cos kx\right) & \text{if } d > 0. \end{cases}$$