CALCULUS II, SUMMER 2015 - EXAM 3 SOLUTIONS

`130 points total = 120 points + 10 extra credit points`

Problem 1 (20 points). Find the radius of convergence of the *hypergeometric series*

$$ {}_2F_1(\alpha,\beta,\gamma;x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha+n-1)!(\beta+n-1)!(\gamma-1)!}{n!(\alpha-1)!(\beta-1)!(\gamma+n-1)!}x^n, $$

where α, β, and γ are positive integers. Compute $x_2F_1(1,1,2;-x)$, and differentiate $x_2F_1(1,1,2;-x)$ term-by-term to figure out which function $x_2F_1(1,1,2;-x)$ converges to.

Solution. The constant term has no bearing on the radius of convergence of ${}_2F_1$, and so it suffices to consider

$$ \sum_{n=1}^{\infty} \frac{(\alpha+n-1)!(\beta+n-1)!(\gamma-1)!}{n!(\alpha-1)!(\beta-1)!(\gamma+n-1)!}x^n. $$

We set

$$ a_n = \frac{(\alpha+n-1)!(\beta+n-1)!(\gamma-1)!}{n!(\alpha-1)!(\beta-1)!(\gamma+n-1)!} $$

and observe that

$$ \begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(\alpha+n)!(\beta+n)!(\gamma-1)!/(n+1)!(\alpha-1)!(\beta-1)!(\gamma+n)!}{(\alpha+n-1)!(\beta+n-1)!(\gamma-1)!/n!(\alpha-1)!(\beta-1)!(\gamma+n-1)!} \\ &= \frac{(a+n)(\beta+n)}{(n+1)(\gamma+n)}. \end{aligned} $$

Therefore,

$$ \lim_{n\to\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1, $$

and so the radius of convergence of the hypergeometric series is

$$ R = \left(\lim_{n\to\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} = 1. $$

Now,

$$ \begin{aligned} x_2F_1(1,1,2,;-x) &= x + x\sum_{n=1}^{\infty} \frac{(n!)(n!)(1!)}{(n!)(0!)(0!)(n+1)!}(-x)^n \\ &= x + x\sum_{n=1}^{\infty} \frac{1}{n+1}(-x)^n = x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}x^{n+1} \\ &= x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n. \end{aligned} $$

Date: August 13, 2015.

Differentiating x_2F_1 term-by-term, we obtain

$$\frac{d}{dx}x_2F_1(1,1,2;-x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1} = \sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1+x}.$$

It therefore follows that

$$x_2F_1(1,1,2;-x) = \int \frac{1}{1+x}\,dx = \ln(1+x) + C.$$

Plugging in $x = 0$, we obtain the identity

$$C = 0.$$

We conclude that

$$x_2F_1(1,1,2;-x) = \ln(1+x).$$

□

Problem 2 (20 points).

(1) Find the Taylor series of

$$f(x) = \frac{1}{3-x}$$

centered at 0, and compute its radius of convergence.

(2) Find the Taylor series of

$$f(x) = \frac{1}{3-x}$$

centered at 2, and compute its radius of convergence.

Solution. (1) Observe that

$$\frac{1}{3-x} = \frac{1/3}{1-x/3} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n.$$

Setting $a_n = \frac{1}{3^{n+1}}$, we see that the radius of convergence is

$$R = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} = \left(\lim_{n\to\infty}\frac{1}{3}\right)^{-1} = 3.$$

(2) Observe that

$$\frac{1}{3-x} = \frac{1}{1-(x-2)} = \sum_{n=0}^{\infty}(x-2)^n.$$

Setting $a_n = 1$, we see that the radius of convergence is

$$R = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} = \left(\lim_{n\to\infty}1\right)^{-1} = 1.$$

□

Problem 3 (20 points). Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{(1.5)_{k+1}}{k!} (-0.75)^k ,$$

where

$$(\alpha)_k = \alpha(\alpha - 1) \cdots (\alpha - k + 1)$$

is the Pochhammer symbol.

Solution. Recall that

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k$$

for each $\alpha \in \mathbb{R}$ and every $x \in (-1, 1)$. Term-by-term differentiation yields

$$\alpha(1 + x)^{\alpha-1} = \sum_{k=1}^{\infty} \frac{k(\alpha)_k}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{(\alpha)_k}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{(\alpha)_{k+1}}{k!} x^k.$$

We now plug in $\alpha = 1.5$ and $x = -0.75$ to conclude that

$$\sum_{k=0}^{\infty} \frac{(1.5)_{k+1}}{k!} (-0.75)^k = 1.5 \cdot (1 - 0.75)^{0.5} = \frac{3}{2}\sqrt{\frac{1}{4}} = \frac{3}{4}.$$

□

Problem 4 (20 points). Show that the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n}$$

converges conditionally.

Solution. Observe that

$$\cos(n\pi/3) = \begin{cases} 1/2 & \text{if } n = 6k + 1 \text{ for some integer } k; \\ -1/2 & \text{if } n = 6k + 2 \text{ for some integer } k; \\ -1 & \text{if } n = 6k + 3 \text{ for some integer } k; \\ -1/2 & \text{if } n = 6k + 4 \text{ for some integer } k; \\ 1/2 & \text{if } n = 6k + 5 \text{ for some integer } k; \\ 1 & \text{if } n = 6k + 6 \text{ for some integer } k. \end{cases}$$

Therefore,

$$\sum_{n=1}^{\infty} \left| \frac{\cos n\pi/3}{n} \right| \geq \sum_{n=1}^{\infty} \frac{1/2}{n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n},$$

and so the comparison test implies that

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi/3)}{n} \right|$$

diverges.

Now,

$$\frac{1}{2} - \frac{1}{2} - 1 - \frac{1}{2} + \frac{1}{2} + 1 = 0,$$

and so
$$\sum_{n=1}^{N} \cos(n\pi/3) = \sum_{n=1}^{N_6} \cos(n\pi/3),$$
where N_6 is the remainder of $N \div 6$. It then follows from the triangle inequality that
$$\left|\sum_{n=1}^{N} \cos(n\pi/3)\right| = \left|\sum_{n=1}^{N_6} \cos(n\pi/3)\right| \leq \sum_{n=1}^{N_6} |\cos(n\pi/3)|.$$
Since $|\cos(n\pi/3)| \geq 0$ for all n, it follows that
$$\sum_{n=1}^{N_6} |\cos(n\pi/3)|. \leq \sum_{n=1}^{6} |\cos(n\pi/3)| = \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{2} + 1,$$
whence we conclude that
$$\left|\sum_{n=1}^{N} \cos(n\pi/3)\right| \leq 4.$$
As $\frac{1}{n} \searrow 0$, the Dirichlet test now implies that
$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n}$$
converges. □

Problem 5 (20 points). Define
$$a_{2n} = (n!)^{-2/n} \text{ and } a_{2n+1} = \frac{1}{n^2}$$
for each $n \geq 0$. Prove that
$$\lim_{n\to\infty} a_n = 0.$$

Solution. Observe that a simple arrangement yields the following:
$$(n!)^2 = (n \cdot 1)((n-1)\cdot 2)((n-2)\cdot 3)\cdots(3\cdot(n-2))(2\cdot(n-1))(1\cdot n).$$
In other words,
$$(n!)^2 = \prod_{k=1}^{n} (n+1-k)k.$$
Now, for each $1 \leq k \leq n$,
$$(n+1-k)k - n = nk - n + k - k^2 = n(k-1) + k(k-1) = (n+k)(k-1) \geq 0,$$
and so
$$(n+1-k)k \geq n.$$
It follows that
$$(n!)^2 = \prod_{k=1}^{n} (n+1-k)k \geq \prod_{k=1}^{n} n = n^n,$$
whence it follows that
$$(n!)^{-2/n} \leq \frac{1}{n}$$
for all $n \in \mathbb{N}$.

Note now that
$$\frac{1}{n^2} \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. We then see that

$$a_{2n} = (n!)^{-2/n} \leq \frac{1}{n} \quad \text{and} \quad a_{2n+1} = \frac{1}{n^2} \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. □

Observe that

$$\frac{1}{n} = \frac{2}{2n} \leq \frac{3}{2n}$$

and that

$$\frac{1}{n} = \frac{2n+1}{n(2n+1)} = \frac{2n+1}{n} \cdot \frac{1}{2n+1} \leq 3 \cdot \frac{1}{2n+1}$$

for all $n \in \mathbb{N}$. It follows that

$$a_{2n} \leq \frac{3}{2n} \quad \text{and} \quad a_{2n+1} \leq \frac{3}{2n+1}$$

for all $n \in \mathbb{N}$, whence

$$a_n \leq \frac{3}{n}$$

for all $n \in \mathbb{N}$.

It now suffices to observe that

$$a_n \geq 0$$

for all $n \in \mathbb{N}$; the desired result follows from the squeeze theorem..

Problem 6 (20 points). Compute the fifth-degree Taylor polynomial $T_5(x)$ of $\tan x$, centered at 0. What is $T_5(1)$?

Hint: $\tan x = \frac{\sin x}{\cos x}$.

Solution. Use long division: see p.486 of the textbook for a detailed solution.

$$T_5(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5,$$

and so

$$T_5(1) = 1 + \frac{1}{3} + \frac{2}{15} = \frac{22}{15}.$$

□

Bonus Problem (10 points). Recall that the *Fourier cosine series* of an even function $f(x)$ on the interval $[-\pi, \pi]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt.$$

Use the fact that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

when $f(x) = |x|$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. Since $|x|$ and $\cos(nx)$ are both even, $|x|\cos(nx)$ is even as well. Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

If $n = 0$, then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi.$$

If $n \geq 1$, then we integrate by substitution and then integrate by parts to obtain the following:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \\ &= \frac{2}{\pi n^2}\int_0^{n\pi} t\cos t\,dt \\ &= \frac{2}{\pi n^2}\left((n\pi)\sin(n\pi) - 0\cdot\sin(0) - \int_0^{n\pi}\sin t\,dt\right) \\ &= -\frac{2}{\pi n^2}\int_0^{n\pi}\sin t\,dt. \end{aligned}$$

Since

$$\int_0^{n\pi}\sin t\,dt = \begin{cases} 2 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even;} \end{cases}$$

we see that

$$a_n = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

It follows that the Fourier cosine series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2}\cos(2n-1)x.$$

Since

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2}\cos(2n-1)x,$$

we see that

$$0 = f(0) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2}.$$

Rearranging, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Now,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

□